# STRUCTURAL PROPERTIES OF MATRIX FRACTION DESCRIPTIONS AND APPLICATIONS IN LINEAR SYSTEMS

BY

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## Declarations

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other C.N.A.A. or University award. None of the material herein has been used in any other submission for an academic award.

A programme of advanced study was undertaken in partial fulfilment of the requirements including guided reading in the theory of $\lambda$ matrices and module theory (under the direction of Dr. A.C. Pugh) and attendance at postgraduate seminars at Plymouth Polytechnic, Cambridge University and the University of Manchester Institute of Science and Technology. A paper entitled "Infinite Pole and Zero Considerations for System Transfer Function Matrices" was presented at the Third Institute of Mathematics and its Applications Conference on Control Theory at the University of Sheffield in September, 1980. Seminars were also presented to the Department of Mathematics, Statistics and Computing and to the Research Association at Plymouth Polytechnic.

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## Abstract

The finite poles and zeros of a rational matrix G(s) are defined to be the zeros of the polynomial denominator and numerator matrices respectively taken from any relatively prime matrix fraction description of G(s). The infinite poles and zeros of G(s) are then defined as the poles and zeros at $w = 0$ of $G(\frac{1}{w})$. This new definition is the central result of the thesis. From it various results for the theory of rational matrices are derived, many of which are analagous to results in the complex variable theory of rational functions. The infinite poles and zeros of a rational matrix are investigated in particular and the definition and results obtained are compared and contrasted with alternative definitions of the infinite poles and zeros as described by other authors.

The relevant results from linear multivariable systems theory are recalled and the results derived for rational matrices in general are applied to the rational transfer function matrix and polynomial system matrix. In particular the infinite decoupling zeros, infinite system poles and zeros and the infinite poles and zeros of the transfer function are investigated. Constant transformations on system matrices which preserve the system characteristics at both finite and infinite frequencies are discussed. A new non-constant transformation on polynomial matrices which preserves both the finite and infinite zeros is described and this is applied to system matrices. Attempts are made to derive a method by which any polynomial system matrix may be realised in the generalised state-space form without any of the finite or infinite system characteristics being altered. Finally the effect of constant output feedback on the decoupling zeros and the transfer function poles and zeros is discussed and multivariable lags, multivariable root locus and the theory of decoupling are briefly considered.

# Chapter 1. Preliminaries

## Section (1.1) : System Equations and System Matrices

Any linear multivariable system may be described (Rosenbrock 1970, Wolovich 1974) by a set of linear differential and algebraic equations of the form

$$T(D)\xi(t) = U(D)u(t) \tag{1.1a}$$

$$y(t) = V(D)\xi(t) + W(D)u(t) \tag{1.1b}$$

where

$$D \equiv \frac{d}{dt} \tag{1.2}$$

and $\xi(t)$, $u(t)$ and $y(t)$ are respectively the $r,\ell$ and $m$ dimensional vectors of system, input and output variables. The matrices $T,U,V$ and $W$ are matrices of polynomials in the differential operator $D$ with coefficients from $\mathbb{C}$, the field of complex numbers, and their dimensions are respectively $r\times r$, $r\times\ell$, $m\times r$ and $m\times\ell$. It is assumed that

$$\left| T(D) \right| \not\equiv 0$$

for otherwise (1.1a) would be indeterminate.

It is more convenient to work in the frequency domain than the time domain for then the problem of solution reduces to being a purely algebraic one. Accordingly, taking Laplace transforms and assuming zero initial conditions equations (1.1) give rise to the purely algebraic equations

$$T(s)\bar{\xi} = U(s)\bar{u} \tag{1.3a}$$

$$\bar{y} = V(s)\bar{\xi} + W(s)\bar{u} \tag{1.3b}$$

where s is the Laplace transformation variable and $\bar{\xi}$, $\bar{u}$ and $\bar{y}$ are the Laplace transforms of the system variable vector $\xi(t)$ the input vector u(t) and the output vector y(t) respectively. The matrices T(s), U(s), V(s) and W(s) are now polynomial matrices in the Laplace transform variable s with coefficients in $\mathbb{C}$.

The equations (1.3) may be written as a single matrix equation

$$\left[\begin{array}{c:c} T(s) & U(s) \\ \hdashline -V(s) & W(s) \end{array}\right] \left[\begin{array}{c} \bar{\xi} \\ \hdashline -\bar{u} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hdashline -\bar{y} \end{array}\right]. \tag{1.4}$$

In this equation the $(r+m) \times (r+\ell)$ polynomial matrix

$$P(s) = \left[\begin{array}{c:c} T(s) & U(s) \\ \hdashline -V(s) & W(s) \end{array}\right] \tag{1.5}$$

contains all the mathematical information required to study the system and thus P(s) is called the POLYNOMIAL SYSTEM MATRIX for the system S. In general no distinction will be made between a system and any description of it and hence, in the rest of this thesis S and P(s) are synonomous.

Many systems can be simply described, after taking Laplace transforms and assuming zero initial conditions, by linear equations of the form

$$(sI-A)\bar{x} = B\bar{u} \tag{1.6a}$$

$$\bar{y} = C\bar{x} + D(s)\bar{u} \tag{1.6b}$$

where $\bar{x}$ is an n-vector (the state), $\bar{u}$ is an $\ell$-vector (the input) and y is an m-vector (the output). The nxn, $n\mathrm{x}\ell$ and mxn matrices, A, B and C are constant and are called the plant matrix, input matrix and output matrix respectively. In general D(s) is a polynomial matrix in s with coefficients in $\mathbb{C}$. Equations (1.6) are said to be in STATE-SPACE FORM and the system matrix in state-space form, corresponding to (1.6) is clearly

$$P(s) = \left[\begin{array}{c:c} sI-A & B \\ \hdashline -C & D(s) \end{array}\right] . \tag{1.7}$$

This is a special case of the polynomial system matrix (1.5). Any system can be described by many different system matrices, all of which contain the important mathematical information required about the system. In particular any system described by a polynomial system matrix could alternatively be described by a different system matrix in state-space form. Clearly analysis of the state-space system matrix is much more straightforward than analysis of the more complicated polynomial system matrix. In order to simplify the analysis therefore various authors (Rosenbrock 1970, Wolovich 1974) have derived algorithms by which any polynomial system matrix can be transformed to the state-space form without any of the important characteristics of the system being lost. This procedure is called state-space realisation. In this thesis the main emphasis will be on the behaviour of systems at infinite frequencies. As will be shown later, although state-space realisation preserves the system characteristics at finite frequencies it does not preserve the behaviour of the system at infinite frequencies and thus alternative procedures will be required in order that the system behaviour at infinite frequencies may also be preserved.

The direct relationship between the output of the system and the input is described by the transfer function matrix $G(s)$. In the case of systems described by a polynomial system matrix $P(s)$ as in (1.5)

$$G(s) = V(s)T^{-1}(s)U(s) + W(s). \tag{1.8}$$

It is clear that $G(s)$ is a matrix of rational functions, i.e. $G(s)$ is a rational matrix. If

$$\lim_{s \to \infty} G(s)$$

exists $G(s)$ is said to be PROPER and if additionally

$$\lim_{s \to \infty} G(s)=0$$

then $G(s)$ is said to be STRICTLY PROPER.

The transfer function matrix of a system in state-space form is given by

$$G(s) = C(sI-A)^{-1}B + D(s). \tag{1.9}$$

The first term on the righthand side of this equation is strictly proper and the second term is polynomial. Consequently $G(s)$ is proper if and only if $D$ is constant and $G(s)$ is strictly proper if and only if $D$ is zero.

Clearly then, one important system characteristic which must be common to all polynomial matrix descriptions of the same system is the transfer function matrix. In the next section more important system characteristics, namely the decoupling zeros and the system poles and system zeros will be discussed.

## Section (1.2) : Some Properties of System Matrices

A square polynomial matrix whose determinant is a non-zero constant is called a unimodular matrix. The inverse of a unimodular matrix is also a unimodular polynomial matrix. Unimodular matrices play an important role in the various matrix equivalence transformations. A very important canonical form for a polynomial matrix is the SMITH FORM which is defined as follows (Rosenbrock 1970):

(2.1):Definition: The Smith form $\Lambda(s)$ of an $m \times \ell$ polynomial matrix $P(s)$ of rank $p$ is given by

$$\Lambda(s) = M(s)P(s)N(s) \tag{2.2}$$

where $M(s)$ and $N(s)$ are unimodular matrices, respectively $m \times m$ and $\ell \times \ell$, and,

$$\Lambda(s) = \begin{bmatrix} Q(s) & 0_{p,\ell-p} \\ 0_{m-p,p} & 0_{m-p,\ell-p} \end{bmatrix} \tag{2.3}$$

where

$$Q(s) = \text{diag}\,(\varepsilon_1(s), \varepsilon_2(s), \;.....,\varepsilon_p(s)) \tag{2.4}$$

and the monic polynomials $\varepsilon_i(s)$ $i = 1,2,....,p$ satisfy the divisibility property

$$\varepsilon_i(s) \;\big|\; \varepsilon_{i+1}(s) \tag{2.5}$$

The $\varepsilon_i(s)$ $i = 1,2,....,p$ are called the INVARIANT POLYNOMIALS of $P(s)$. The invariant polynomials $\varepsilon_i(s)$ can be factorised into their monic irreducible factors $\phi_j(s)$ over the field of the coefficients. Let the power of $\phi_j$ occurring in $\varepsilon_i(s)$ be $k_{ij}$. Then those $\phi_j^{k_{ij}}$ having

$k \neq 0$ are called the ELEMENTARY DIVISORS OF P(s) (Rosenbrock (1970)).

There are various methods by which the Smith form of a polynomial matrix can be determined. Since the operations of premultiplying or postmultiplying a polynomial matrix by a unimodular matrix, as in equation (2.2), simply amount to performing simple row or column operations respectively on that matrix the Smith form of any polynomial matrix P(s) can be found by performing simple row and column operations on P(s) until a diagonal matrix in which the diagonal elements satisfy (2.5) is achieved.

Alternatively, the greatest common divisor $D_i(s)$ of all the minors of order i, i = 1,2,....,p can be found. By definition $D_o(s) = 1$. Then the $\varepsilon_i(s)$ are given by

$$\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)} \qquad i = 1,2,\ldots,p$$

and hence the Smith form may be constructed. The $D_i(s)$, i = 1,2,....,p are called (Gantmacher, 1959) the DETERMINANTAL DIVISORS of P(s).

Suppose that $P(s_o)$ has rank less than p for some $s_o \in \mathbb{C}$. Then P(s) is said to have a ZERO at $s_o$ and in fact $s_o$ is a zero of some invariant polynomial in the Smith form of P(s). i.e. $(s-s_o)^k$ is an elementary divisor of P(s) for some positive integer k. If $s_o$ is a zero of P(s) then there exist polynomial matrices L(s) and $P_1(s)$ such that

$$P(s) = L(s)P_1(s) \qquad (2.6)$$

where

$$\left| L(s) \right| = (s-s_o)^k \tag{2.7}$$

for some positive integer k and the rank of $P_1(s_o)$ is p.

Now let

$P(s) = (T(s)\ U(s))$

and

$P_1(s) = (T_1(s)\ U_1(s))$.

If equation (2.6) holds L(s) is said to be a COMMON LEFT DIVISOR of T(s) and U(s). The Smith form of a unimodular matrix is an appropriately dimensioned identity matrix. Clearly then a unimodular matrix has no zeros as defined here.. Consequently, if every left divisor of T(s) and U(s) is a unimodular matrix then T(s) and U(s) are said to be RELATIVELY LEFT PRIME. The zeros of rational matrices will be considered in much greater detail later in this thesis and in this work relatively prime polynomial matrices will play a very important role. The following very important result concerning relatively prime polynomial matrices forms part of theorem 6.1 in Rosenbrock (1970).

<u>(2.8):Theorem</u>: The polynomial matrices T(s), U(s) respectively $r \times r$, $r \times \ell$ are relatively left prime if and only if one of the following equivalent conditions is satisfied

i) The rank of $(T(s)\ U(s))$ is r for all $s \in \mathbb{C}$.

ii) The Smith form of $(T(s)\ U(s))$ is $(I_r\ 0)$.

Similarly, the polynomial matrices $T(s)$, $V(s)$ respectively $r \times r$, $m \times r$ are said to be RELATIVELY RIGHT PRIME if and only if every COMMON RIGHT DIVISOR of $T(s)$ and $V(s)$, such as $R(s)$, where

$$\begin{bmatrix} T(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} T_2(s) \\ V_2(s) \end{bmatrix} R(s)$$

is unimodular. An analogous result to theorem (2.8) holds for relatively right prime polynomial matrices.

Returning to the system matrix

$$P(s) = \left[\begin{array}{c:c} T(s) & U(s) \\ \hdashline -V(s) & W(s) \end{array}\right] \qquad (2.9)$$

with its associated transfer function matrix

$$G(s) = V(s)T^{-1}(s)U(s) + W(s) \qquad (2.10)$$

the decoupling zeros of the system are defined as follows:

(2.11): Definition: An element $\beta \in \mathbb{C}$ is called an INPUT-DECOUPLING ZERO (i.d. zero) of $P(s)$ in case $\beta$ is a zero of the matrix $(T(s)\ U(s))$. The set $\{\beta_1, \beta_2, \ldots\ldots \beta_b\}$ of all such zeros is called the set of INPUT-DECOUPLING ZEROS of $P(s)$, i.e. $\{\beta_j\}$ is the set of i.d. zeros of $P(s)$.

The set of OUTPUT-DECOUPLING ZEROS (o.d. zeros) $\{\gamma_j\}$ is defined in an entirely analogous manner as the set of zeros of the matrix $\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix}$

The set of input-decoupling zeros which are at the same time output-decoupling zeros is called the set of INPUT-OUTPUT DECOUPLING ZEROS (i.o.d. zeros) of $P(s)$ and is denoted by $\{\delta_j\}$.

The set $\{\beta_j, \gamma_j\} - \{\delta_j\}$ is called the set of DECOUPLING ZEROS of P(s).

The following theorem provides for the absence of decoupling zeros.

<u>(2.12):Theorem</u>: P(s) has no i.d. zeros if and only if T(s) and U(s) are relatively left prime. P(s) has no o.d. zeros if and only if T(s) and V(s) are relatively right prime.

Now consider the systems

$$P(s) = \left[\begin{array}{c:c} L_1(s)T_1(s) & L_1(s)U_1(s) \\ \hdashline -V(s) & W(s) \end{array}\right] \qquad (2.13)$$

and

$$P_1(s) = \left[\begin{array}{c:c} T_1(s) & U_1(s) \\ \hdashline -V(s) & W(s) \end{array}\right] \qquad (2.14)$$

where $T_1(s)$ and $U_1(s)$ are relatively left prime and $L_1(s)$ is not in general unimodular although $|L_1(s)| \not\equiv 0$. Thus the i.d. zeros of P(s) are the zeros of $L_1(s)$ and $P_1(s)$ has no i.d. zeros. Now the transfer function of P(s) is given by

$$\begin{aligned} G(s) &= V(s)(L_1(s)T_1(s))^{-1}L_1(s)U_1(s) + W(s) \\ &= V(s)T_1^{-1}(s)L_1^{-1}(s)L_1(s)U_1(s) + W(s) \\ &= V(s)T_1^{-1}(s)U_1(s) + W(s). \end{aligned}$$

Clearly then P(s) and $P_1(s)$ have the same transfer function matrix and the i.d. zeros of P(s) have been cancelled out or "decoupled" in the formation of the transfer function matrix. This explains the term "decoupling zeros".

Thus it is seen that for a given transfer function matrix $G(s)$ there are many system matrices giving rise to it. Such system matrices are called REALISATIONS of $G(s)$.

The order n of a system is given by the degree of $|T(s)|$. Obviously, unless $L_1(s)$ is unimodular in which case $P(s)$ has no i.d. zeros, the order of $P(s)$ is greater than the order of $P_1(s)$. As each i.d. zero is removed from $P(s)$ a system of lower order results. There is clearly no upper bound on the order of a realisation of a given transfer function matrix $G(s)$ since the addition of more decoupling zeros increases the order of a realisation but has no effect on $G(s)$. There is, however, a lower bound on the order of a realisation of $G(s)$, denoted by $\nu(G)$. Any realisation with order $n = \nu(G)$ is said to have LEAST ORDER. Rosenbrock (1970) gives the following characterisation of such realisations.

<u>(2.15):Theorem</u>: A system matrix $P(s)$ has least order if and only if one of the following equivalent conditions holds.

(i) $P(s)$ has no decoupling zeros.

(ii) $T(s)$, $U(s)$ are relatively left prime, and
$T(s)$, $V(s)$ are relatively right prime.

Since the least order $\nu(G)$ is a characteristic of a given rational matrix $G(s)$ it is desirable to have a method of computing $\nu(G)$ directly from $G(s)$ rather than indirectly from realisations of it. One such method was described by Rosenbrock (1970,p117).

(2.16):Theorem: The least order $\nu(G)$ of the rational matrix $G(s)$ is the degree of the least common denominator of all minors of all orders of $G(s)$.

Two further important system characteristics are the system poles and the system zeros. The poles were defined thus by Rosenbrock (1970,p66):

(2.17):Definition: The set of POLES OF THE SYSTEM $P(s)$, denoted $\{n_i\}$ is the set of zeros of $|T(s)|$.

Rosenbrock (1974b) defined the system zeros in terms of certain minors of $P(s)$.

(2.18):Definition: Let $P(s)$ be an $(r+m)\times(r+\ell)$ polynomial system matrix as in equation (1.5). Then the minor formed from rows $1,2,\ldots r,r+i_1,\ldots r+i_k$ and columns $1,2\ldots,r,r+j_1,\ldots,r+j_k$ of $P(s)$ is denoted by

$$P\begin{pmatrix} i_1,i_2,\ldots,i_k \\ j_1,j_2,\ldots,j_k \end{pmatrix}$$

and called a BORDERED MINOR OF $P(s)$ OF ORDER $k$.

Thus

(2.19):Definition: The SYSTEM ZEROS $\{\alpha_i\}$ are the zeros of the greatest common divisor of the bordered minors of $P(s)$ of order $p$, i.e. of the form

$$P\begin{pmatrix} i_1,i_2,\ldots i_p \\ j_1,j_2,\ldots j_p \end{pmatrix} \tag{2.20}$$

where $p$ is the greatest integer for which $P(s)$ possesses a non-zero minor of the form (2.20).

In this section the important characteristics of a system, namely the decoupling zeros and the system poles and system zeros have been defined. The relationship between these characteristics will be described in section (4.1). In the next section transformations of the system equations which leave unchanged these system characteristics will be discussed.

## Section (1.3) : Equivalence of System Matrices

Any system S may be described by many different sets of equations of the form (1.1). When given a complicated set of equations it is often necessary to reduce them to a simpler form before the system can be analysed. This simpler form may of course be the state-space form of equations (1.6). One advantage of using the system matrix idea is that such transformations of the system equations may be represented as transformations on the system matrix P(s) which are more easily understood. The transformations on P(s) which are important are those which do not change the transfer function matrix G(s) or the decoupling zeros of P(s) for then such transformations do not effectively change the system or its important characteristics. A well-known transformation of this type is strict system equivalence which was described by Rosenbrock (1970). Let

$$P(s) = \left[\begin{array}{c:c} T(s) & U(s) \\ \hdashline -V(s) & W(s) \end{array}\right] \quad \text{and} \quad P_1(s) = \left[\begin{array}{c:c} T_1(s) & U_1(s) \\ \hdashline -V_1(s) & W_1(s) \end{array}\right] \qquad (3.1)$$

be two $(r+m)\times(r+\ell)$ polynomial system matrices. Then

(3.2) :Definition: $P(s)$ and $P_1(s)$ are said to be STRICTLY SYSTEM EQUIVALENT (written (s.s.e)) if and only if there exist rxr unimodular matrices $M(s)$ and $N(s)$ and polynomial matrices $X(s)$ and $Y(s)$ of dimensions (mxr) and (rxℓ) respectively such that

$$\begin{bmatrix} M(s) & \vdots & 0 \\ \cdots & & \cdots \\ X(s) & \vdots & I_m \end{bmatrix}\begin{bmatrix} T(s) & \vdots & U(s) \\ \cdots & & \cdots \\ -V(s) & \vdots & W(s) \end{bmatrix}\begin{bmatrix} N(s) & \vdots & Y(s) \\ \cdots & & \cdots \\ 0 & \vdots & I_\ell \end{bmatrix} = \begin{bmatrix} T_1(s) & \vdots & U_1(s) \\ \cdots & & \cdots \\ -V_1(s) & \vdots & W_1(s) \end{bmatrix}. \tag{3.3}$$

In general the order n of $P(s)$, defined as the degree of $|T(s)|$, bears no relation to the dimension r of $T(s)$. However, it is normally required that

$$r \geq n \tag{3.4}$$

for otherwise $P(s)$ cannot be reduced to its state-space form by operations of (s.s.e.) alone.

Some important consequences of definition (3.2) are

(3.5) :Theorem: Under strict system equivalence the following are invariant

i) The transfer function matrix $G(s)$ and consequently the set of finite poles $\{g_i\}$ of $G(s)$ and $\nu(G)$ their number.

ii) The sets $\{\beta_i\}$, $\{\gamma_i\}$ and $\{\delta_i\}$ of finite i.d., o.d. and i.o.d. zeros respectively of $P(s)$.

iii) The set of decoupling zeros $\{\beta_i,\gamma_i\} - \{\delta_i\}$ of $P(s)$.

iv) The set $\{n_i\}$ of zeros of $|T(s)|$ and n their number.

Proof: See Rosenbrock (1970).

If operations of s.s.e. are carried out on a system matrix in state-space form, in general, the result will not be in state-space form. Because of the importance of the state-space form, a transformation which preserves this form is required and the relevant definition is

(3.6) : Definition: Two system matrices P(s) and $P_1(s)$ in state-space form are said to be SYSTEM SIMILAR (written (s.s.)) if and only if there exists a constant non-singular matrix H such that,

$$\begin{bmatrix} H^{-1} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_m \end{bmatrix}\begin{bmatrix} sI_n - A & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D(s) \end{bmatrix}\begin{bmatrix} H & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & I_\ell \end{bmatrix} = \begin{bmatrix} sI_n - A_1 & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} . \quad (3.7)$$

Thus (s.s.) is a special case of (s.s.e.) and clearly the results of theorem (3.5) also hold for system similarity. The relationship between these definitions is (Rosenbrock 1970)

(3.8) : Theorem: Two system matrices in state-space form are (s.s.) if and only if they are (s.s.e.).

One disadvantage of (s.s.e.) is that a polynomial system matrix is not (s.s.e.) to a trivial expansion of itself e.g. The system matrices $P_1(s)$ and $P_2(s)$ where

$$P_1(s) = \begin{bmatrix} s & \vdots & 1 \\ \cdots & & \cdots \\ 1 & \vdots & 0 \end{bmatrix} \quad (3.9)$$

and

$$P_2(s) = \begin{bmatrix} s & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 0 \\ \cdots & \cdots & & \cdots \\ 1 & 0 & \vdots & 0 \end{bmatrix} \quad (3.10)$$

are not (s.s.e.) even though they clearly possess the same transfer function matrix and the same decoupling zero structure.

This problem has been researched by Pernebo (1977), Furhrmann (1977) and Pugh and Shelton (1978). By allowing trivial expansion in addition to the operations of (s.s.e.) this problem is solved, and also the requirement (3.4) is removed.

Strict system equivalence is essentially the definition of equivalence of polynomial matrices (Gantmacher 1959, Lancaster 1970) which Pugh and Shelton refer to as unimodular equivalence (written (u.e.)) applied to system matrices. Pugh and Shelton define a new equivalence transformation for polynomial matrices which they term extended unimodular equivalence (written (e.u.e.)). They then apply this transformation to system matrices giving a new equivalence transformation called extended strict system equivalence.. Extended unimodular equivalence is defined as follows:

(3.11) :Definition: Let $P_1(s)$ and $P_2(s)$ be two polynomial matrices of dimensions $(r_1+m)\times(r_1+\ell)$ and $(r_2+m)\times(r_2+\ell)$ respectively. Then $P_1(s)$ and $P_2(s)$ are said to be EXTENDED UNIMODULAR EQUIVALENT in case there exist polynomial matrices $M(s)$ and $N(s)$ such that

$$M(s)P_1(s) = P_2(s)N(s) \tag{3.12}$$

where $M(s)$ and $P_2(s)$ are relatively left prime and $P_1(s)$ and $N(s)$ are relatively right prime.

The following results, given by Pugh and Shelton are important consequences of these definitions.

(3.13) : Theorem: Two polynomial matrices of the same dimensions that are (e.u.e.) are also (u.e.).

(3.14) : Theorem: If the polynomial matrices $P_1(s)$ and $P_2(s)$ are (e.u.e.) then so are their respective Smith forms $S(P_1)$ and $S(P_2)$. More particularly

$$\left.\begin{array}{lr} S(P_2) = I_{r_2-r_1} \oplus S(P_1) & r_2 > r_1 \\ S(P_2) = S(P_1) & r_2 = r_1 \\ I_{r_1-r_2} \oplus S(P_2) = S(P_1) & r_1 > r_2 \end{array}\right\} \qquad (3.15)$$

(3.16) : Theorem: If $S(P_1)$ and $S(P_2)$ are related as in (3.15) then $P_1(s)$ and $P_2(s)$ are (e.u.e.).

Theorems (3.14) and (3.16) show that two polynomial matrices are (e.u.e.) if and only if the Smith form of one is identical to or a trivial expansion of the Smith form of the other. This shows that any operations that can be achieved by (e.u.e.) can also be achieved by the operations of trivial expansion and (u.e.) and vice versa.

Pugh and Shelton apply the transformation of (e.u.e.) to system matrices as follows.

(3.17) : Definition: Let $P(s)$ and $P_1(s)$ be two polynomial system matrices of dimensions $(r+m) \times (r+\ell)$ and $(r_1+m) \times (r_1+\ell)$ respectively. Then $P(s)$ and $P_1(s)$ are said to be EXTENDED STRICT SYSTEM EQUIVALENT (written(e.s.s.e.)) if and only if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I_m \end{bmatrix} \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} = \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I_\ell \end{bmatrix} \tag{3.18}$$

where $M(s)$ and $T_1(s)$ are relatively left prime and $T(s)$ and $N(s)$ are relatively right prime, i.e. $T(s)$ and $T_1(s)$ are (e.u.e.).

This transformation is identical to that proposed by Fuhrmann (1977) and it has also been discussed by Kailath (1980 Chapter 8). Note that the requirement that $T(s)$ and $T_1(s)$ are (e.u.e.) is a sufficient but not a necessary condition for $P(s)$ and $P_1(s)$ also to be (e.u.e.). The following theorem is an important consequence of definition (3.17).

<u>(3.19) :Theorem</u>: The transformation of (e.s.s.e.) preserves

i) the transfer function matrix and hence its least order $\nu(G)$.

ii) the set of system poles.

iii) all sets of decoupling zeros.

iv) the set of system zeros.

<u>Proof</u>: See Pugh and Shelton (ibid).

The importance of (e.s.s.e.) is that it incorporates trivial expansion in addition to the usual operations of (s.s.e.) as the next result states.

<u>(3.20) :Theorem</u>: Trivial expansion is an operation of (e.s.s.e.).

<u>Proof</u>: See Pugh and Shelton (ibid).

Returning to equations (3.9) and (3.10), $P(s)$ and $P_1(s)$ are related by

$$\left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} s & 1 \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} s & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right]. \qquad (3.21)$$

It is clear from (3.21) that $P(s)$ and $P_1(s)$ are (e.s.s.e.).

Pugh and Shelton also prove the following theorem.

(3.22) : Theorem: Any polynomial system matrix $P(s)$ may be reduced to state-space form by operations of (e.s.s.e.).

This result is important because it shows that given a polynomial system matrix one can always find a state-space representation of it whether or not $r \geqslant n$. If originally $r > n$ the state-space system will have greater dimensions than the original system.

The importance of the transformations of (e.u.e.) and (e.s.s.e.) will become more clear in the rest of this thesis.

# Chapter 2. The Finite and Infinite Zeros and Poles of a Rational Matrix.

## Section (2.1) : The Basic Definitions

The finite poles and zeros of a rational matrix have been defined by many authors (e.g. Rosenbrock 1970) and a summary was given in Chapter 1. More recently attention (Rosenbrock 1974a, Verghese, Kailath et al. 1979, 1981, Anderson and Bitmead 1978) has been focussed on defining the infinite poles and zeros of a rational matrix. In this section new definitions of both the finite and infinite poles and zeros of a rational matrix will be presented. These definitions are motivated by well-known techniques from the complex variable theory of rational functions. Much of the material on which this section and the next two sections are based is taken from Pugh and Ratcliffe (1979a).

Before considering rational matrices the zeros of a polynomial matrix will be defined. This procedure is adopted so that the closest possible analogy to the case of rational functions, as indicated below, is obtained. Accordingly, let D(s) be a polynomial matrix of dimension $m \times \ell$, then

(1.1):Definition: $s_0 \varepsilon \mathbb{C}$ is a FINITE ZERO OF DEGREE k of D(s) in case $(s-s_0)^k$ is an elementary divisor of D(s). The set of ZEROS of D(s) is the set of all such numbers $s_0$, a zero of degree k being included k times.

This is essentially Rosenbrock's (1974b) definition wherein the zeros of D(s) are the zeros of the invariant polynomials of D(s) taken all together. The only new concept in (1.1) is that of the degree of a zero. The following result yields a simple test to determine whether $s_o$ is a zero of D(s) although it yields no information concerning the degree.

<u>(1.2):Proposition</u>: $s_o \varepsilon \mathbb{C}$ is a zero of the polynomial matrix D(s) if and only if

$$\text{RANK } D(s_o) < \rho(D)$$

where $\rho(\ )$ denotes the normal rank of the indicated matrix.

In complex variable theory the finite poles and zeros of a rational function g(s) are defined by way of a factorisation

$$g(s) = \frac{n(s)}{d(s)}$$

where n(s) and d(s) are relatively prime polynomial functions. The zeros and poles of g(s) are defined as the zeros of n(s) and d(s) respectively. An analogous method will be used here to define the finite poles and zeros of a rational matrix G(s), hence in the first instance factorisation of rational matrices into relatively prime polynomial factors will be discussed.

It is well known that any $m \times \ell$ rational matrix G(s) may be decomposed into relatively prime polynomial factors

$$G(s) = N(s)D^{-1}(s) = D_1^{-1}(s)N_1(s) \qquad (1.3)$$

where $N(s)$, $D(s)$ are relatively right prime and $N_1(s)$, $D_1(s)$ are relatively left prime, i.e. $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ and $(N_1(s), D_1(s))$ have full column and row rank respectively for all $s \in \mathbb{C}$.

Of course neither of the factorisations (1.3) is unique since for any unimodular matrix $M(s)$ or $M_1(s)$

$$D'(s) = D(s)M(s) \qquad N'(s) = N(s)M(s)$$

and

$$D_1'(s) = M_1(s)D_1(s) \qquad N_1'(s) = M_1(s)N_1(s)$$

also form relatively prime factorisations of $G(s)$. Despite this however the following natural terminology, due in part to Wolovich (1973) is adopted.

(1.4):Definition: Any $m \times \ell$ polynomial matrix such as $N(s)$ or $N_1(s)$ satisfying (1.3) is called a NUMERATOR of $G(s)$.

(1.5):Definition: Any $m \times m$ polynomial matrix such as $D_1(s)$ or $\ell \times \ell$ matrix such as $D(s)$ satisfying (1.3) is called a DENOMINATOR of $G(s)$.

Since, as indicated above, there are many relatively prime factorisations of a given rational matrix $G(s)$ the numerator and denominator are not unique. Thus the natural definition of poles and zeros in terms of zeros of numerators and denominators cannot immediately be made since it is not clear for example whether different denominators even have the same zeros. This situation is clarified by the following results, the first of which establishes the complete connection between the different relatively prime polynomial factorisations of $G(s)$.

(1.6): Theorem: All numerators of $G(s)$ are unimodular equivalent while all denominators are extended unimodular equivalent.

Proof: If $N_1(s)$ and $N_2(s)$ are two numerators of $G(s)$ and $D_1(s)$, $D_2(s)$ their respective denominators in any relatively left prime factorisation of $G(s)$

i.e. $G(s) = D_1^{-1}(s)N_1(s) = D_2^{-1}(s)N_2(s)$

then the result follows from Rosenbrock (1970 p.139). In fact $D_1(s)$ and $D_2(s)$ are actually unimodular equivalent which is a special case of extended unimodular equivalence (Pugh and Shelton, 1978). The theorem is true in the same way in the case of two relatively right prime factorisations.

i.e. $G(s) = N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s)$

If the factorisations are of opposite type

$$G(s) = N_1(s)D_1^{-1}(s) = D_2^{-1}(s)N_2(s) \qquad (1.7)$$

i.e. (as above)

then

$$D_2(s)N_1(s) = N_2(s)D_1(s) \qquad (1.8)$$

However, from the relative primeness of the factorisations (1.7), the relation (1.8) is simply the statement that $N_1(s)$ and $N_2(s)$ are extended unimodular equivalent, as indeed are $D_1(s)$ and $D_2(s)$. Since $N_1(s)$ and $N_2(s)$ are both of the same dimensions, $m \times \ell$, the relationship of extended unimodular equivalence can actually be replaced by one of unimodular equivalence (Pugh and Shelton 1978) which proves the theorem.

In the above theorem the result concerning the numerators was originally proved by Wolovich (1973), although as indicated in the proof, certain special cases are attributed to Rosenbrock (1970). The complete result of theorem (1.6) is inherent in the work of Fuhrmann (1977) but the simple proof given here is a direct consequence of the results of Pugh and Shelton (1978).

The importance of the result (1.6) to the present discussion is made clear by theorem (1.3.14) which forms the central result of Pugh and Shelton (1978). Since the Smith forms of extended unimodular equivalent matrices differ only by trivial expansion it is seen that all denominators (respectively numerators), possess the same non-trivial, non-unit invariant polynomials. Accordingly the following corollary to theorem (1.6) can be stated.

<u>(1.9):Corollary:</u> All numerators (denominators) of a rational matrix G(s) have the same set of zeros. Specifically, if $s_o$ is a zero of degree k of a numerator (denominator) of G(s) it is a zero of degree k of every numerator (denominator).

As a direct consequence of this corollary it is now possible to formally define the poles and zeros of a rational matrix in the manner proposed. Thus

<u>(1.10):Definition:</u> $s_o \varepsilon \mathbb{C}$ is a ZERO OF DEGREE k of the rational matrix G(s) if it is a zero of degree k of any numerator of G(s). $s_o \varepsilon \mathbb{C}$ is a POLE OF DEGREE k of the rational matrix G(s) if it is a zero of degree k of any denominator of G(s).

This definition of the finite poles and zeros of G(s) is essentially the same as that given by Rosenbrock (1970, p.109) as will now be seen. However, the concept of the degree of a pole or zero is new. Rosenbrock (ibid) investigates the finite poles and zeros of the $m \times \ell$ rational matrix G(s) by means of the McMillan standard form which is defined as follows:

<u>(1.11):Definition</u>: Let d(s) be the monic least common denominator of all the elements of G(s) and write

$$G(s) = N(s)/d(s). \tag{1.12}$$

N(s) can be brought to Smith form by the transformation

$$L(s)N(s)R(s) = S(s) \tag{1.13}$$

where L(s) and R(s) are unimodular matrices. In the rational matrix S(s)/d(s) there may be common factors between numerator and denominator in the elements on the leading diagonal. After cancellation of these common factors the rational matrix M(s) results where

$$M(s) = \begin{cases} (Q(s),\ 0_{m,\ell-m}) & m < \ell \\ Q(s) & m = \ell \\ \begin{bmatrix} Q(s) \\ 0_{m-\ell,\ell} \end{bmatrix} & m > \ell \end{cases} \tag{1.14}$$

$$\text{and } Q(s) = \operatorname{diag}\left[\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \ldots \frac{\varepsilon_p(s)}{\psi_p(s)}, 0, \ldots, 0\right] \tag{1.15}$$

where p is the normal rank of G(s) and $\varepsilon_i | \varepsilon_{i+1}$, $\psi_{i+1} | \psi_i$, $i=1, \ldots, p-1$. Then M(s) is the McMILLAN FORM of G(s).

The following corollary establishes a result that will be required later in this section.

(1.16):Corollary: $\psi_1(s)$, as defined in equation (1.15) is equal to d(s), the monic least common denominator of all the elements of G(s).

Proof: Let $\varepsilon_1'(s)$ denote the first element on the leading diagonal of S(s). From the definition of the Smith form of a polynomial matrix (1.2.1) $\varepsilon_1'(s)$ is the greatest common divisor of all the elements of N(s). Since, by definition, d(s) is the monic least common denominator of all the elements of G(s), no polynomial factor of d(s) is also a common factor of all the elements of N(s). Consequently $\varepsilon_1'(s)$ and d(s) are relatively prime, and, from the construction of M(s)

$$\varepsilon_1(s) = \varepsilon_1'(s)$$

and, more importantly

$$\psi_1(s) = d(s).$$

Rosenbrock (ibid) defines the finite poles and zeros of G(s) via the McMillan form of G(s) in the following way:

(1.17):Definition: The FINITE ZEROS of G(s) are the zeros of the $\varepsilon_i(s)$ and the FINITE POLES of G(s) are the zeros of the $\psi_i(s)$ each counted according to their multiplicity and degree.

The definitions (1.10) and (1.17) are reconciled by the next theorem in which the relationship between numerators and denominators of G(s) and the McMillan standard form of G(s) is established.

(1.18): Theorem: Let

$$E(s) = \begin{cases} (\varepsilon(s), \ 0_{m,\ell-m}) & m < \ell \\ \varepsilon(s) & m = \ell \\ \begin{bmatrix} \varepsilon(s) \\ 0_{m-\ell,\ell} \end{bmatrix} & m > \ell \end{cases} \qquad (1.19)$$

where $\varepsilon(s) \equiv \mathrm{diag}\,(\varepsilon_1(s), \varepsilon_2(s), \ \ldots, \ \varepsilon_p(s), \ 0, \ \ldots, \ 0)$
and let

$$K(s) = \mathrm{diag}\,(\psi_1(s), \psi_2(s), \ \ldots, \ \psi_p(s), 0, \ldots, 0) \qquad (1.20)$$

where $K(s)$ is $\ell \times \ell$ and the $\varepsilon_i(s)$ and $\psi_i(s)$ are as defined in equations (1.14) and (1.15). Then all numerators of $G(s)$ are unimodular equivalent to $E(s)$ and all denominators of $G(s)$ are extended unimodular equivalent to $K(s)$.

Proof: Let

$$G(s) = N_1(s) D_1^{-1}(s) \qquad (1.21)$$

be a relatively right prime factorisation of $G(s)$.
From (1.12) and (1.13)

$$\begin{aligned} M(s) &= L(s)\ G(s)\ R(s) \qquad (1.22) \\ &= E(s) K^{-1}(s) \end{aligned}$$

from (1.19) and (1.20). This is a relatively right prime factorisation of $M(s)$ and consequently $E(s)$ and $K(s)$ are respectively a numerator and denominator of $M(s)$.

Substituting for G(s) from (1.21) into (1.22) gives

$$M(s) = L(s)N_1(s)D_1^{-1}(s)R(s)$$
$$= L(s)N_1(s)(R^{-1}(s)D_1(s))^{-1}.$$

This is also a relatively right prime factorisation of M(s) since L(s) and R(s) (and consequently $R^{-1}(s)$) are unimodular. Therefore $L(s)N_1(s)$ and $R^{-1}(s)D_1(s)$ are also a numerator and a denominator of M(s) respectively.

Since $R^{-1}(s)$ is unimodular $D_1(s)$ is unimodular equivalent to $R^{-1}(s)D_1(s)$. But $R^{-1}(s)D_1(s)$ and K(s) are both denominators taken from relatively right prime factorisations of M(s) and hence they are also unimodular equivalent. Consequently, by the transitivity of unimodular equivalence, (Pugh and Shelton, 1978, Kailath, 1980) $D_1(s)$ and K(s) are unimodular equivalent. Thus, by theorem (1.6) all denominators of G(s) are extended unimodular equivalent to K(s).

The result for the numerators follows similarly, the relationship of extended unimodular equivalence being replaced by one of unimodular equivalence since E(s) and all numerators of G(s) all have the same dimensions, $m \times \ell$.

The result for the numerators is well known and was noted by Pugh and Shelton (1978) and Kailath (1980) but the result for the denominators is new. From this result it is clear that

(1.23):Corollary:Definitions (1.10) and (1.17) are equivalent i.e. they define the same sets of zeros and poles although (1.17) makes no mention of the degree of a pole or zero.

Theorem (1.18) and Corollary (1.16) lead immediately to the following result which will be required in the next section.

(1.24): Theorem: Let D(s) be any denominator of G(s) and let D(s) have Smith form

$$S(D) = \begin{bmatrix} \phi_r(s) & 0 & \cdots & 0 \\ 0 & \phi_{r-1}(s) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \phi_1(s) \end{bmatrix}$$

where $r = \ell$ or $m$ depending on the factorisation of G(s). Then

$$\phi_1(s) = d(s)$$

where d(s) is the monic least common denominator of all the elements of G(s).

Proof: From (1.20) K(s) has Smith form

$$S(K) = \begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & 1 & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ & & & 1 & & & \vdots \\ & & & & \psi_p & & \vdots \\ & & & & & \psi_{p-1} & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & & & & 0 & \psi_1 \end{bmatrix}$$

By theorem (1.18), D(s) and K(s) are extended unimodular equivalent and hence, by theorem (1.3.14) they have the same Smith form up to trivial expansion.

Consequently

$$\phi_i(s) = 1 \qquad \text{for } i = r\text{-}p, \ldots, r-1, r$$

and $\quad \phi_i(s) = \psi_i(s) \qquad$ for $i = 1, 2, \ldots, p$.

In particular $\phi_1(s) = \psi_1(s)$ but, from Corollary (1.16) $\psi_1(s) = d(s)$ and thus

$\phi_1(s) = d(s)$ as required.

Having defined the finite poles and zeros of a rational matrix it is now required to define the terms "infinite poles" and "infinite zeros". In order to define these in a way which again preserves the direct analogy with the scalar case of a rational function the standard technique of complex variable theory is utilised, whereby the simple bilinear transformation $s = \frac{1}{w}$ is performed. This transformation takes the point $s = \infty$ to the point $w = 0$ and the point $s = 0$ to the point $w = \infty$. All other points in the complex s-plane are carried onto finite points in the complex w-plane in a one-to-one manner. Thus, as was also suggested by Verghese et al (1981), this transformation and definition (1.10) can be combined to give

<u>(1.25):Definition</u>:G(s) is said to have an INFINITE ZERO OF DEGREE k in case $w = 0$ is a finite zero of degree k of the rational matrix $G(\frac{1}{w})$.

G(s) is said to have an INFINITE POLE OF DEGREE k in case $w = 0$ is a finite pole of degree k of $G(\frac{1}{w})$.

The following example illustrates the main definitions which have been described in this section.

<u>(1.26):Example</u>:Consider the rational matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & s^2 \\ 0 & (s+2)^2 \end{bmatrix}$$

Now

$$G(s) = \begin{bmatrix} 1 & s^2 \\ 0 & (s+2)^2 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= N(s)\, D^{-1}(s)$$

This is a relatively right prime factorisation since

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} 1 & s^2 \\ 0 & (s+2)^2 \\ s+1 & 0 \\ 0 & 1 \end{bmatrix}$$

has full column rank for all $s \in \mathbb{C}$. Now N(s) and D(s) have Smith forms

$$\begin{bmatrix} 1 & 0 \\ 0 & (s+2)^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}$$

respectively. Hence, by definition (1.10), G(s) has one finite zero of degree 2 at $s = -2$ and one finite pole of degree 1 at $s = -1$.

The finite poles and zeros of G(s) can, of course, also be found by examining the McMillan form of G(s). Applying the method and notation of definition (1.11) to this example gives

$d(s) = s+1$.

Hence

$$N(s) = \begin{bmatrix} 1 & s^2(s+1) \\ 0 & (s+2)^2(s+1) \end{bmatrix}$$

and N(s) has Smith form

$$S(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+2)^2(s+1) \end{bmatrix}.$$

Dividing each of the elements of S(s) by d(s) and cancelling the common factors gives the McMillan form of G(s)

$$M(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & (s+2)^2 \end{bmatrix}$$

where $\varepsilon_1 = 1$

$\varepsilon_2 = (s+2)^2$

$\psi_1 = s+1$

$\psi_2 = 1.$

Note that $\psi_1 = d(s)$ as predicted by Corollary (1.16). Thus, according to definition (1.17), G(s) has a finite zero at $s = -2$ and a finite pole at $s = -1$.

Hence it is clear that definitions (1.10) and (1.17) define the same sets of poles and zeros although definition (1.17) makes no mention of the degree of a pole or zero. This example illustrates Corollary (1.23).

In order to investigate the infinite poles and zeros of G(s) the transformation $s = \frac{1}{w}$ is made giving

$$G(\tfrac{1}{w}) = \begin{bmatrix} \frac{w}{1+w} & \frac{1}{w^2} \\ 0 & (\frac{1+2w}{w})^2 \end{bmatrix}.$$

$G(\frac{1}{w})$ can be factorised as

$$G(\tfrac{1}{w}) = \begin{bmatrix} w & 1 \\ 0 & (1+2w)^2 \end{bmatrix} \begin{bmatrix} 1+w & 0 \\ 0 & w^2 \end{bmatrix}^{-1}$$

$$= \tilde{N}(w)\tilde{D}^{-1}(w).$$

This is a relatively right prime factorisation of $G(\frac{1}{w})$ since

$$\begin{bmatrix} \tilde{N}(w) \\ \tilde{D}(w) \end{bmatrix} = \begin{bmatrix} w & 1 \\ 0 & (1+2w)^2 \\ 1+w & 0 \\ 0 & w^2 \end{bmatrix}$$

has full column rank for all finite w. Now $\tilde{N}(w)$ and $\tilde{D}(w)$ have Smith forms

$$\begin{bmatrix} 1 & 0 \\ 0 & w(1+2w)^2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & w^2(1+w\ ) \end{bmatrix}$$

respectively. Hence $G(\frac{1}{w})$ has one zero of degree 1 at $w = 0$ and one pole of degree 2 at $w=0$. Consequently, by definition (1.25) G(s) has one infinite zero of degree 1 and one infinite pole of degree 2.

## Section (2.2): The Poles of a Rational Matrix - Further Results

In this section more of the consequences of the definitions given in section (2.1) for the poles of a rational matrix will be explored and certain of the results in complex variable theory concerning rational functions will be generalised to rational matrices.

The following is one such result.

(2.1): Theorem: $s_o \varepsilon \mathbb{C}$ is a pole of the rational matrix G(s) if and only if, for some i and j,

$$\lim_{s \to s_o} g_{ij}(s) = \infty \quad (2.2)$$

Proof: If $s_o \varepsilon \mathbb{C}$ is a pole of the rational matrix G(s) then $(s-s_o)$ is a factor of some invariant polynomial of the denominators of G(s). In particular, if the Smith form of the denominator of G(s) is

$$S(D) = \mathrm{diag}(\emptyset_r, \emptyset_{r-1}, \ \ldots, \ \emptyset_1)$$

where $r = \ell$ or m depending on the factorisation of G(s), then $(s-s_o)$ divides $\emptyset_i(s)$ for some i. By the divisibility properties of the $\emptyset_i(s)$ $(s-s_o)$ must be a factor of $\emptyset_1(s)$. But, by theorem (2.1.24), $\emptyset_1$ is the monic least common denominator of all the elements of G(s) and hence $(s-s_o)$ occurs in the denominator of at least one element $g_{ij}(s)$ of G(s). Thus, by the corresponding theorem of complex variable theory

$$\lim_{s \to s_o} g_{ij}(s) = \infty.$$

Conversely, since $g_{ij}(s)$ is a rational function, its only singularities are poles (finite or infinite). Thus, if (2.2) holds for finite $s_o$, then $s_o$ is a finite pole of $g_{ij}(s)$. Consequently $(s-s_o)$ is a factor of $\emptyset_1(s)$ and hence $s_o$ is a pole of G(s).

The next result which, approached from a different point of view, forms exercise (4.1) in Rosenbrock (1970, p.113), although not important on its own, will be required in the proof of the theorem that follows it.

(2.3): Theorem: Let

$$G(s) = A(s) + B(s)$$

where $G(s)$ is a rational matrix and $B(s)$ is polynomial. Then $A(s) + B(s)$ and $A(s)$ have precisely the same set of finite poles.

Proof: Let

$$A(s) = P^{-1}(s)Q(s) \tag{2.4}$$

be a relatively left prime factorisation of $A(s)$. Thus

$$\begin{aligned} G(s) &= P^{-1}(s)\, Q(s) + B(s) \\ &= P^{-1}(s)\, (Q(s) + P(s)B(s)) \end{aligned} \tag{2.5}$$

This is a polynomial factorisation of $G(s)$ but it is not yet clear whether or not it is a relatively left prime factorisation. Now consider the matrix

$$(P(s),\ Q(s) + P(s)B(s)) = (P(s),\ Q(s)) \begin{bmatrix} I_m & B(s) \\ 0 & I_\ell \end{bmatrix} . \tag{2.6}$$

Since (2.4) is a relatively left prime factorisation of $A(s)$ the matrix $(P(s), Q(s))$ has full row rank. Also the matrix

$$\begin{bmatrix} I_m & B(s) \\ 0 & I_\ell \end{bmatrix}$$

is unimodular and consequently, from (2.6) the matrix

$$(P(s),\ Q(s) + P(s)B(s))$$

must also have full row rank. Hence (2.5) is a relatively left prime factorisation of $G(s)$.

Thus it can be seen from (2.4) and (2.5) that $P(s)$ is a denominator of both $A(s)$ and $G(s)$ and therefore, from definition (1.10), $A(s)$ and $A(s) + B(s)$ have the same set of finite poles.

The following theorem, which generalises another well known result concerning rational functions, allows the computation involved in the calculation of the finite and infinite poles of a rational matrix to be greatly reduced. However, it is the corollaries that follow the theorem that are particularly important since they will be referred to many times later in this thesis.

(2.7):Theorem: Let $G(s)$ be an $m \times \ell$ rational matrix. Then $G(s)$ may be written as

$$G(s) = G_s(s) + D(s) \tag{2.8}$$

where $G_s(s)$ is strictly proper and $D(s)$ is polynomial. Then

(i) the finite poles of $G(s)$ are the finite poles of $G_s(s)$

and

(ii) the infinite poles of $G(s)$ are the infinite poles of $D(s)$.

Proof: (i) This follows immediately from (2.1) since $G_s(s)$ is a special case of $A(s)$.

(ii) Substituting $s = \frac{1}{w}$ in (2.8) gives

$$G(\tfrac{1}{w}) = G_s(\tfrac{1}{w}) + D(\tfrac{1}{w}). \tag{2.9}$$

Now let

$$G_s(\tfrac{1}{w}) = \frac{1}{d(w)}M(w) \tag{2.10}$$

where $d(w)$ is the monic least common denominator of all the elements of $G_s(\frac{1}{w})$ and $M(w)$ is a polynomial matrix. Now since $G_s(s)$ is strictly proper each element $g_{ij}(s)$ of $G_s(s)$ is a strictly proper rational function

i.e. $\lim\limits_{s \to \infty} g_{ij}(s) = 0$

for all i and j. Thus in each $g_{ij}(s)$ the degree of the numerator is less than the degree of the denominator. Consequently, in $g_{ij}(\frac{1}{w})$, w is not a factor in the denominator. Hence w is not a factor in d(w), i.e. $d(0) \neq 0$.

Substituting for $G_S(\frac{1}{w})$ from (2.10) in (2.9) gives

$$G(\tfrac{1}{w}) = \frac{1}{d(w)} M(w) + D(\tfrac{1}{w})$$

$$= \frac{1}{d(w)} (M(w) + d(w)D(\tfrac{1}{w})).$$

Now the infinite poles of G(s) are the poles at w=0 of $(M(w) + d(w)D(\frac{1}{w}))$, which are, by theorem (2.3) the poles at w=0 of $d(w)D(\frac{1}{w})$, since M(w) is polynomial. But since $d(0) \neq 0$ the poles at w=0 of $d(w)D(\frac{1}{w})$ are simply those of $D(\frac{1}{w})$. Hence the poles at w=0 of $D(\frac{1}{w})$ are the poles at w=0 of $G(\frac{1}{w})$ and clearly the infinite poles of D(s) are exactly those of G(s) as required.

<u>(2.11):Corollary</u>: The $m \times \ell$ rational matrix G(s) is polynomial if and only if it has no finite poles.

<u>Proof</u>: G(s) may be expanded as in (2.8) to give

$$G(s) = G_S(s) + D(s)$$

where in this case $G_S(s)$ is the $m \times \ell$ null matrix. Hence, by theorem (2.7), the finite poles of G(s) are the poles of the null matrix, i.e. G(s) has no finite poles.

Another way of proving this result, which does not depend on theorem (2.7), is to let G(s) have a relatively left prime factorisation

$$G(s) = I_m^{-1} G(s).$$

Hence $I_m$ is a denominator of G(s), and clearly this has no finite zeros, i.e. G(s) has no finite poles.

Conversely, suppose that

$$G(s) = P^{-1}(s)Q(s)$$

is a relatively left prime factorisation of G(s). If G(s) has no finite poles then all the denominators of G(s) have no finite zeros. Thus the Smith form of D(s) is $I_m$, and so D(s) is unimodular. Consequently $D^{-1}(s)$ is a polynomial matrix as is G(s).

<u>(2.12):Corollary</u>: A non-constant polynomial matrix has all its poles at infinity.

<u>Proof</u>: If G(s) is a non-constant polynomial matrix then it must have at least one non-constant polynomial element.

i.e. $g_{ij}(s) = a_k s^k + a_{k-1}s^{k-1} + \ldots\ldots + a_1 s + a_o \ , \ a_k \neq 0$

for some i and j. Thus

$$g_{ij}(\tfrac{1}{w}) = \frac{a_k}{w^k} + \frac{a_{k-1}}{w^{k-1}} + \ldots\ldots + \frac{a_1}{w} + a_o$$

$$= \frac{1}{w^k}\,(a_k + a_{k-1}w + \ldots\ldots + a_1 w^{k-1} + a_o w^k).$$

Thus

$$\lim_{w \to 0} g_{ij}(\tfrac{1}{w}) = \infty$$

and hence, by theorem (2.1), $G(\frac{1}{w})$ has a pole at w = 0 i.e. G(s) has at least one pole at infinity.

From corollary (2.11) it is obvious that G(s) has no finite poles and hence G(s) has all its poles at infinity.

Corresponding results hold for proper matrices as the following corollaries show.

(2.13): Corollary: A rational matrix $G(s)$ is proper if and only if it has no infinite poles.

Proof: By theorem (2.7), $G(s)$ may be expanded as

$$G(s) = G_s(s) + D(s)$$

where in this case $D(s)$ is a constant matrix since $G(s)$ is proper. But the infinite poles of $G(s)$ are the infinite poles of $D(s)$ and consequently, since $D(s)$ clearly has no infinite poles, $G(s)$ has no infinite poles.

Conversely, suppose that $G(s)$ has no infinite poles, then $G(\frac{1}{w})$ has no poles at $w = 0$. Thus $w$ is not a factor of any invariant polynomial of the denominators of $G(\frac{1}{w})$. In particular $w$ is not a factor in $\tilde{\phi}_1(w)$, the last such invariant polynomial. But $\tilde{\phi}_1(w)$ is the least common denominator of elements of $G(\frac{1}{w})$ and therefore $w$ is not a factor in the denominator of any $g_{ij}(\frac{1}{w})$.

Thus, for all i and j

$$g_{ij}(\tfrac{1}{w}) = \frac{a_\tau + a_{\tau-1}w + \dots + a_o w^\tau}{b_\rho + b_{\rho-1}w + \dots + b_o w^\rho} \tag{2.14}$$

where $b_{\rho(i,j)} \neq 0$ and $a_{o(i,j)} \neq 0$.

Putting $w = \frac{1}{s}$ gives

$$g_{ij}(s) = \frac{\frac{1}{s^\tau}\,(a_\tau s^\tau + a_{\tau-1}s^{\tau-1} + \dots + a_o)}{\frac{1}{s^\rho}\,(b_\rho s^\rho + b_{\rho-1}s^{\rho-1} + \dots + b_o)}$$

$$= \frac{a_\tau s^\rho + a_{\tau-1}s^{\rho-1} + \dots\dots + a_o s^{\rho-\tau}}{b_\rho s^\rho + b_{\rho-1}s^{\rho-1} + \dots\dots + b_o} \qquad \rho>\tau$$

A similar rational function results if $\tau \geqslant \rho$. In all cases since $a_o \neq 0$ and $b_\rho \neq 0$ the degree of the numerator of $g_{ij}(s)$ is at most equal to the degree of the denominator of $g_{ij}(s)$. Thus $g_{ij}(s)$ is a proper rational function for all i and j. i.e. G(s) is a proper rational matrix.

<u>(2.15):Corollary</u>: A rational matrix G(s) has infinite poles if and only if it is non-proper.

<u>Proof</u>: The proof follows immediately from corollary (2.13).

The final corollary in this section extends theorem (2.1) to the case of infinite poles.

<u>(2.16):Corollary</u>: G(s) has a pole at infinity if and only if for some i and j

$$\lim_{s \to \infty} g_{ij}(s) = \infty.$$

<u>Proof</u>: G(s) has a pole at infinity if and only if it is non-proper by corollary (2.15). But by definition G(s) is non-proper if and only if for some i and j

$$\lim_{s \to \infty} g_{ij}(s) = \infty$$

and the corollary follows immediately.

The definitions given in section (2.1) and the results developed subsequently allow a new interpretation of a special type of rational matrix namely a unimodular matrix.

Unimodular matrices were defined in section (1.2) and were seen to play an important role in various equivalence transformations on polynomial matrices. In the following theorem a unimodular matrix G(s) is shown to be characterised by its lack of poles and zeros at finite values of s.

(2.17):Theorem: If G(s) is a full rank mxm matrix then it is unimodular if and only if it has no finite poles and no finite zeros.

Proof: If G(s) has no finite poles then, by corollary (2.11) it is polynomial. Since the full rank polynomial matrix G(s) has no finite zeros its Smith form is $I_m$. i.e. G(s) is unimodular.

Conversely, if G(s) is polynomial then it has no finite poles. Also, since G(s) is unimodular,

$$G(s) = \left[G(s)^{-1}\right]^{-1} I_m$$

is a relatively left prime factorisation of G(s) from which it is clear that G(s) has no finite zeros.

Unimodular matrices will be discussed further in the next section.

Section (2.3): The McMillan Degree and Related Results.

Rosenbrock (1970) has also defined the infinite poles and zeros of an $m \times \ell$ rational matrix in the following way:

(3.1):Definition: (i) If any element of G(s) tends to infinity as $s \to \infty$ then G(s) is said to have a pole at infinity.
(ii) If every minor of some given order k tends to zero as $s \to \infty$ then G(s) is said to have a zero at infinity.

As can be seen from corollary (2.16) this definition of infinite poles corresponds to that given in (1.25). However, while (3.1)(ii) may be a sufficient condition for an infinite zero to exist, it is not a necessary condition as will be shown below.

Rosenbrock (ibid) has given an alternative characterisation of infinite poles and zeros which was originally formulated by McMillan (1952).

(3.2):Definition: Let

$$s = \frac{\alpha p}{p-1} \tag{3.3}$$

where $\alpha$ is a constant which is not a finite pole or zero of a minor of any order of G(s). Then G(s) has an infinite pole if and only if $G(\frac{\alpha p}{p-1})$ has a pole at p = 1 in the sense of definition (1.16) and hence definition (1.10). Similarly G(s) has an infinite zero in case $G(\frac{\alpha p}{p-1})$ has a zero at p = 1 in the sense of definition (1.16). This definition is independent of $\alpha$ providing that $\alpha$ is not a pole or zero of any minor of any order of G(s).

Note that the transformation (3.3) does not include $s = \frac{1}{w}$. The following example illustrates the various definitions.

(3.4):Example: Let

$$G(s) = \begin{bmatrix} s & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$$

Now

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$$

is a relatively left prime factorisation and hence, by definition (1.10), G(s) has a finite pole and a finite zero at s = 0, both of degree one.

In the case of the infinite poles and zeros

$$G(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w} & 0 \\ 0 & w \end{bmatrix}$$

$$= \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}$$

is a relatively left prime factorisation. Hence, by definition (1.25), G(s) has one infinite pole and one infinite zero, both of degree one.

Note that G(s) does have one element which tends to infinity as $s \to \infty$ and so (3.1)(i) predicts the existence of at least one pole at infinity. This is, of course, consistent with the findings of corollary (2.16). On the contrary however (3.1)(ii) does not predict the existence of the infinite zero for G(s) since the only 2x2 minor of G(s) is unity and not all 1x1 minors of G(s) tend to zero as $s \to \infty$.

To illustrate definition (3.2) make the substitution (3.3) where $\alpha \neq 0$. Then

$$G(\tfrac{\alpha p}{p-1}) = \begin{bmatrix} \frac{\alpha p}{p-1} & 0 \\ 0 & \frac{p-1}{\alpha p} \end{bmatrix} \qquad (3.5)$$

The Smith McMillan form of (3.5) is then

$$\begin{bmatrix} \frac{1}{p(p-1)} & 0 \\ 0 & p(p-1) \end{bmatrix}$$

from which it is clear that $G(\frac{\alpha p}{p-1})$ has both a pole and a zero at p=1. Thus, by definition (3.2), G(s) has both a pole and a zero at infinity.

It is clear from the above example that (3.1)(ii) is not a necessary condition for G(s) to possess one or more infinite zeros and hence definition (3.1) is invalid. It is noted that the results concerning definitions (1.25) and (3.2) are similar despite the fact that the transformation (3.3) does not include $s = \frac{1}{w}$. However, since both are bilinear transformations, it would seem natural that they should lead to the same results.

In fact, the equivalence of definitions (1.25) and (3.2) may be proved directly.

<u>(3.6):Theorem</u>: The definitions (1.25) and (3.2) are equivalent.

<u>Proof</u>: It will be shown that the poles and zeros of $G(\frac{1}{w})$ at $w = 0$ occur in an identical manner to those of $G(\frac{\alpha p}{p-1})$ at $p = 1$.

Let a minor of G(s) of some order be

$$\phi(s) = \frac{a_q s^q + a_{q-1} s^{q-1} + \ldots\ldots + a_1 s + a_o}{b_q s^q + b_{q-1} s^{q-1} + \ldots\ldots + b_1 s + b_o} \qquad (3.7)$$

where the numerator and denominator have no common factors and $a_q$, $b_q$ are not simultaneously zero.

Let $\theta_1(w)$ denote the identical minor to $\phi(s)$ formed from $G(\frac{1}{w})$ and $\theta_2(p)$ denote that formed from $G(\frac{\alpha p}{p-1})$. Then

$$\theta_1(w) = \phi(\tfrac{1}{w})$$

$$= \frac{a_q + a_{q-1}w + \ldots\ldots + a_1 w^{q-1} + a_o w^q}{b_q + b_{q-1}w + \ldots\ldots + b_1 w^{q-1} + b_o w^q} \qquad (3.8)$$

where the numerator and denominator have no common factors since $s = \frac{1}{w}$ is one-to-one.

Similarly

$$\theta_2(p) = \phi(\tfrac{\alpha p}{p-1})$$

$$= \frac{a_q(\alpha p)^q + a_{q-1}(\alpha p)^{q-1}(p-1) + \ldots\ldots + a_1\alpha p(p-1)^{q-1} + a_o(p-1)^q}{b_q(\alpha p)^q + b_{q-1}(\alpha p)^{q-1}(p-1) + \ldots\ldots + b_1\alpha p(p-1)^{q-1} + b_o(p-1)^q} \qquad (3.9)$$

where the numerator and denominator again have no common factors since $s = \dfrac{\alpha p}{p-1}$ is also one-to-one.

From (3.8) and (3.9) it is clear that $\theta_1(w)$ has a zero of degree k (respectively pole of degree k) at $w = 0$ if and only if $\theta_2(p)$ has a zero of degree k (respectively pole of degree k) at $p = 1$.

Thus a determinantal divisor (as defined in section (1.2)) of $G(\frac{1}{w})$ will possess a factor of the form $w^h$, where h is an integer, if and only if the corresponding determinantal divisor of $G(\frac{\alpha p}{p-1})$ possesses a factor of the form $(p-1)^h$. In view of this any invariant polynomial of $G(\frac{1}{w})$ possesses a factor of the form $w^h$ if and only if the corresponding invariant polynomial of $G(\frac{\alpha p}{p-1})$ possesses a factor of the form $(p-1)^h$, which proves the theorem.

As a consequence of this theorem it is clear that various results which Rosenbrock derived, using his definitions of the poles and zeros of a rational matrix, will also hold for the poles and zeros of a rational matrix defined according to definitions (1.10) and (1.25). In particular these definitions permit the usual interpretations of two important concepts in linear systems theory, namely the least order and the McMillan degree of a rational matrix. Most of these results have been discussed by other authors (e.g. McMillan 1952, Rosenbrock ibid, Pugh 1977a). The concept of least order was discussed in section (1.2). The following important results may now be added.

(3.10):Theorem: The least order of a rational matrix $G(s)$, denoted by $\nu(G)$, is equal to the total number of finite poles of $G(s)$, counted according to their multiplicity and degree.

Proof: Let

$$G(s) = T^{-1}(s)U(s)$$

be a relatively left prime factorisation of $G(s)$ so that the poles of $G(s)$, from definition (1.10), are the zeros of $T(s)$. However

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -I & 0 \end{bmatrix}$$

is clearly a least order realisation of $G(s)$. Thus, from (1.2.16)

$$\begin{aligned} \nu(G) &= \text{degree of } |T(s)| \\ &= \text{total number of finite zeros of } T(s) \text{ counted} \\ &\quad\ \text{according to their multiplicity and degree} \\ &= \text{total number of finite poles of } G(s) \end{aligned}$$

as required.

(3.11):Corollary: Let $D(s)$ be a polynomial matrix. Then $\nu(D) = 0$.

Proof: This follows immediately since, by corollary (2.11), $D(s)$ has no finite poles.

(3.12):Corollary: The total number of infinite poles of the polynomial matrix $D(s)$ is equal to $\nu(D(\frac{1}{w}))$.

Proof: Since $D(s)$ is polynomial, by corollary (2.12), it has all its poles at infinity. Hence $D(\frac{1}{w})$ has its poles at $w = 0$. Therefore

$$\begin{aligned}\nu(D(\tfrac{1}{w})) &= \text{total number of poles at } w = 0 \text{ of } D(\tfrac{1}{w}) \\ &= \text{total number of infinite poles of } D(s).\end{aligned}$$

Rosenbrock (ibid) has defined the McMillan degree of a rational matrix $G(s)$ as follows

(3.13):Definition: The McMillan degree of $G(s)$, denoted by $\delta(G)$, is defined by

$$\delta(G) = \nu(G(\frac{\alpha p}{p-1}))$$

where $\alpha$ satisfies the conditions of definition (3.2).

The next important result is a consequence of this definition and theorem (3.10).

(3.14):Theorem: $\delta(G)$ represents the total number of poles, both finite and infinite, of $G(s)$, counted according to their multiplicity and degree.

Proof: In definition (3.2) $\alpha$ was chosen so that it was not equal to any pole or zero of $G(s)$. Hence the bilinear transformation

$$s = \frac{\alpha p}{p-1}$$

maps all the points in the complex s-plane onto finite points in the complex p-plane. i.e. $G(\frac{\alpha p}{p-1})$ has no infinite poles and thus $\nu(G(\frac{\alpha p}{p-1}))$ is equal to the total number of poles of $G(s)$ counted according to their multiplicity and degree, and this is in turn equal to $\delta(G)$.

Clearly $\delta(G)$ is independent of the choice of $\alpha$, subject only to the given conditions, and $\delta(G)$ is unchanged by the bilinear transformations of definitions (1.25) and (3.2). Note however that $\nu(G(\frac{\alpha p}{p-1}))$ is not necessarily equal to $\nu(G(\frac{1}{w}))$. This is because the bilinear transformation

$$s = \frac{1}{w}$$

maps the point $s = 0$ onto the point $w = \infty$. Hence any poles of $G(s)$ at $s = 0$ become infinite poles in $G(\frac{1}{w})$ and consequently $\nu(G(\frac{1}{w}))$ is not in this case equal to the total number of poles of $G(\frac{1}{w})$.

<u>(3.15):Corollary</u>: If $G_s(s)$ is a proper rational matrix, then

$$\delta(G_s) = \nu(G_s).$$

<u>Proof</u>: This follows immediately since, by corollary (2.13), $G_s(s)$ has no infinite poles.

<u>(3.16):Corollary</u>: If $D(s)$ is polynomial, then

$$\delta(D) = \nu(D(\tfrac{1}{w})).$$

<u>Proof</u>: This result follows immediately from corollary (3.12).

Recall theorem (2.7) in which $G(s)$ was written as

$$G(s) = G_s(s) + D(s)$$

where $G_s(s)$ was strictly proper and $D(s)$ was polynomial. In view of theorem (2.7) and theorem (3.14) and its corollaries it is clear that

(3.17):Theorem: The McMillan degree of a rational matrix $G(s)$ is given by $\delta(G(s)) = \nu(G_s(s)) + \nu(D(s^{-1}))$.

This result was originally noted by Kalman (1965) and forms the basis of his definition of degree.

In the case of the scalar rational function $g(s)$ there is a well known interplay between the ideas of poles and zeros of $g(s)$ and $\frac{1}{g(s)}$. It is therefore of interest to know if this situation persists in the case of rational matrices. As a first result in this direction the following theorem due to Rosenbrock (1970) is stated.

(3.18):Theorem: If $G(s)$ is square and non-singular over the field of rational functions then

$$\delta(G) = \delta(G^{-1}).$$

Thus (3.18) indicates that in the case of a square invertible $G(s)$ the total number of poles of $G(s)$ is equal to the total number of poles of $G^{-1}(s)$. In fact more than this can be said, and the following result is in some ways a generalisation of Desoer and Schulman (1974, theorem 4).

(3.19):Theorem: If $G(s)$ is square and invertible then the finite (respectively infinite) poles of $G(s)$ are the finite (respectively infinite) zeros of $G^{-1}(s)$ and the finite (respectively infinite) zeros of $G(s)$ are the finite (respectively infinite) poles of $G^{-1}(s)$.

Proof: Let $G(s)$ have Smith-McMillan form

$$S(G) = \begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\varepsilon_m(s)}{\psi_m(s)} \end{bmatrix}$$

where $\varepsilon_i | \varepsilon_{i+1}$ , $\psi_{i+1} | \psi_1$ for $i = 1,2, \ldots m-1$.

Thus,

$$G(s) = L(s)S(G)R(s)$$

where $L(s)$ and $R(s)$ are unimodular matrices and so

$$G^{-1}(s) = R^{-1}(s)\,(S(G))^{-1}L^{-1}(s).$$

Thus $G^{-1}(s)$ and $(S(G))^{-1}$ have the same Smith-McMillan form namely

$$S(G^{-1}) = \begin{bmatrix} \frac{\psi_m(s)}{\varepsilon_m(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\psi_1(s)}{\varepsilon_1(s)} \end{bmatrix}.$$

This proves the finite case of the theorem.

In the case of the poles and zeros at infinity an analagous argument may be carried out concerning the matrix $G(\frac{1}{w})$ noting that

$$(G(\tfrac{1}{w}))^{-1} = G^{-1}(\tfrac{1}{w})$$

to yield the theorem.

(3.20):Corollary: If G(s) is square and invertible then the total number of poles of G(s) is equal to the total number of zeros.

Proof: This follows directly from theorems (3.18) and (3.19).

In theorem (2.16) a unimodular matrix was seen to have all its poles and zeros at infinity. The next result, which is an extension of corollary (3.20) to the special case of unimodular matrices shows that such matrices have as many infinite poles as they have infinite zeros. This result was also described by Vardulakis (1980).

(3.21):Theorem: If G(s) is a unimodular matrix then

$$\begin{bmatrix}\text{Total number of infinite}\\ \text{poles of } G(s)\end{bmatrix} = \begin{bmatrix}\text{Total number of infinite}\\ \text{zeros of } G(s)\end{bmatrix}$$

$$= \delta(G(s)).$$

Proof: Since G(s) is square and invertible, by corollary (3.20),

$$\begin{bmatrix}\text{Total number of poles}\\ \text{of } G(s)\end{bmatrix} = \begin{bmatrix}\text{Total number of zeros}\\ \text{of } G(s)\end{bmatrix}$$

$$= \delta(G)$$

from (3.14). The theorem now follows immediately since G(s) has all its poles and zeros at infinity.

Corollary (3.20) does not extend to the cases when G(s) is either (i) a full rank non-square rational matrix or (ii) a square non-invertible rational matrix as the next two examples show.

(3.22): Example: Let

$$G(s) = \begin{bmatrix} \frac{s^2}{s-1} & s & 1 \\ 1 & 0 & \frac{s^2}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} s^2 & s(s-1) & s-1 \\ s-1 & 0 & s^2 \end{bmatrix}.$$

From this relatively left prime factorisation of G(s) it is easy to see that G(s) has two finite poles of degree one at s=1 and no finite zeros.

Putting $s={}^1/w$ gives

$$G(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w-w^2} & \frac{1}{w} & 1 \\ 1 & 0 & \frac{1}{w-w^2} \end{bmatrix}$$

$$= \begin{bmatrix} w(1-w) & 0 \\ 0 & w(1-w) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1-w & w(1-w) \\ w(1-w) & 0 & 1 \end{bmatrix}$$

which is relatively left prime factorisation of $G(\frac{1}{w})$. Hence $G(\frac{1}{w})$ has two finite poles of degree one at w=0 and no finite zeros, i.e. G(s) has two infinite poles of degree one and no infinite zeros.

Thus G(s) has four poles, each of degree one, and no zeros and it is clear that corollary (3.20) does not hold in the case of this non-square rational matrix.

The next example deals with the case of square, non-invertible matrices.

(3.23):Example: Let

$$G(s) = \begin{bmatrix} s^2-1 & s \\ \frac{s^2-1}{s^2} & \frac{1}{s} \end{bmatrix}$$

G(s) is clearly non-invertible since $|G(s)| = 0$

Now

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix}^{-1} \begin{bmatrix} s^2-1 & s \\ s^2-1 & s \end{bmatrix}$$

is a relatively left prime factorisation of G(s). Hence G(s) has one finite pole of degree 2 at s = 0 and no finite zeros.

Putting $s = \frac{1}{w}$ gives

$$G(\tfrac{1}{w}) = \begin{bmatrix} \frac{1-w^2}{w^2} & \frac{1}{w} \\ 1-w^2 & w \end{bmatrix}$$

$$= \begin{bmatrix} w^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1-w^2 & w \\ 1-w^2 & w \end{bmatrix}.$$

From this relatively left prime factorisation of $G(\frac{1}{w})$ it is clear that G(s) has one infinite pole of degree 2 and no infinite zeros.

Therefore, in this case, G(s) has two poles each of degree 2 and no zeros and once again corollary (3.20) does not hold.

Thus it can be seen that corollary (3.20) which deals with the case of square invertible rational matrices cannot be extended to either the case of non-square or non-invertible rational matrices. Hence it is not yet clear how, if at all, the total number of poles of a rational matrix is related to the total number of zeros if that matrix is not both square and invertible. This problem has been resolved by Verghese et al (1979) and will be discussed further in section (3.2).

## Chapter 3 : Further Results for Rational Matrices

### Section (3.1): The Infinite Zeros of a Polynomial Matrix

It is well known that a rational function cannot simultaneously have poles and zeros at any $s_o \varepsilon \mathbb{C}$ nor indeed at infinity. Thus a polynomial function, which must have at least one infinite pole, has no infinite zeros. In the case of rational matrices however it is perfectly possible that $s_o \varepsilon \mathbb{C}$ may be both a pole and a zero. In fact there may be poles and zeros of differing degrees at $s_o$. Recall example (2.1.26) in which G(s) had one infinite zero of degree one and one infinite pole of degree two. In particular it is possible that a polynomial matrix may have infinite zeros. The main result in this section will provide for their absence. The first result however, which will be required in the proof of the main result, gives an alternative interpretation of the McMillan degree of a polynomial matrix. This result forms theorem (1.3.41) in Pugh (1977a) and is based on the results of Rosenbrock (1970).

<u>(1.1):Theorem</u>: If G(s) is a polynomial matrix, then $\delta(G)$ is the highest degree among minors of all orders of G(s).

<u>Proof</u>: If G(s) is polynomial then, from (2.3.16)

$$\delta(G) = \nu(G(\tfrac{1}{w}))$$

$$= \text{degree of the least common denominator of minors of all orders of } G(\tfrac{1}{w})$$

from theorem (1.2.16).

Since G(s) is polynomial, the denominators of all elements of $G(\frac{1}{w})$ are of the form $w^q$ for some $q \geqslant 0$. Thus the degree of the denominator of any minor of $G(\frac{1}{w})$ is simply the degree of the corresponding minor of $G(s)$. Accordingly the degree of the least common denominator of all minors of $G(\frac{1}{w})$ is just the highest degree among minors of all orders of $G(s)$.

The following important result specifies the conditions under which a full rank polynomial matrix has no infinite zeros.

<u>(1.2):Theorem</u>: Let $G(s)$ be an $m\times\ell$ polynomial matrix of full rank. $G(s)$ has no infinite zeros if and only if there exists a high order minor ($m\times m$ or $\ell\times\ell$ whichever is the less) of $G(s)$ with degree $\delta(G)$.

<u>Proof</u>: Assume that $m\leq\ell$, the other cases may be proved similarly. Since $G(s)$ is polynomial it has no finite poles, by corollary (2.2.11). Also, by theorem (1.1), $G(s)$ has a minor of degree $\delta(G)$ and from (2.3.16)

$$\delta(G) = \nu(G(\tfrac{1}{w})) \tag{1.3}$$

Let

$$G(\tfrac{1}{w}) = \tilde{D}^{-1}(w)\tilde{N}(w) \tag{1.4}$$

be a relatively left prime factorisation of $G(\frac{1}{w})$, then by (1.3)

$$\delta(|\tilde{D}(w)|) = \delta(G) \tag{1.5}$$

where $|\ |$ denotes the determinant of the indicated matrix.

Suppose $G(s)$ has an $m\times m$ minor of degree $\delta(G)$. Denote this minor by

$$G_{(j_1,j_2,\ldots,j_m)}(s)$$

thus indicating the columns from which it is formed.

Now,

$$G_{(j_1,j_2,\ldots,j_m)}(s)\Big|_{s=\frac{1}{w}} = \left|\tilde{D}^{-1}(w)\right| . \tilde{N}_{(j_1,j_2,\ldots,j_m)}(w)$$

$$= \frac{1}{w^{\delta(G)}} \cdot \tilde{N}_{(j_1,j_2,\ldots,j_m)}(w).$$

Hence

$$\tilde{N}_{(j_1,j_2,\ldots,j_m)}(w) = w^{\delta(G)} \; G_{(j_1,j_2,\ldots,j_m)}(s)\Big|_{s=\frac{1}{w}} . \qquad (1.6)$$

Now

$$G_{(j_1,j_2,\ldots,j_m)}(s) = p_{\delta(G)}s^{\delta(G)} + \ldots + p_1 s + p_o \qquad (1.7)$$

with

$$p_{\delta(G)} \neq 0$$

and so

$$G_{(j_1,j_2,\ldots,j_m)}(s)\Big|_{s=\frac{1}{w}} = \frac{p_{\delta(G)} + \ldots + p_1 w^{\delta(G)-1} + p_o w^{\delta(G)}}{w^{\delta(G)}} . \qquad (1.8)$$

Thus from (1.6) and (1.8)

$$\tilde{N}_{(j_1,j_2,\ldots,j_m)}(w) = p_{\delta(G)} + \ldots + p_1 w^{\delta(G)-1} + p_o w^{\delta(G)} . \qquad (1.9)$$

Since $p_{\delta(G)} \neq 0$ it follows that w does not divide $\tilde{N}_{(j_1,j_2,\ldots,j_m)}(w)$ and so it does not divide the greatest common divisor of the mxm minors of $\tilde{N}(w)$. But this greatest common divisor is just the product of the invariant polynomials of $\tilde{N}(w)$ and so $\tilde{N}(w)$ has no elementary divisors of the form $w^q$ $(q>0)$ i.e. $\tilde{N}(w)$ has no zeros at $w = 0$ i.e. $G(s)$ has no infinite zeros.

Conversely suppose that G(s) has no infinite zeros, then if (1.4) is a prime factorisation of $G(\frac{1}{w})$,

$$\text{Rank } \tilde{N}(0) = m$$

since $m \le l$. Consequently there exists some mxm minor of $\tilde{N}(w)$, say $\tilde{N}_{(j_1,j_2,\ldots,j_m)}(w)$ which is not divisible by w.

i.e. $$\tilde{N}_{(j_1,j_2,\ldots,j_m)}(w) = n_\tau w^\tau + \ldots + n_1 w + n_0 \qquad (1.10)$$

for some $\tau$ and $n_0 \neq 0$. (1.11)

Now since the factorisation (1.4) is prime it follows that

$$\delta(|\tilde{D}(w)|) = \delta(G)$$

and so,

$$G_{(j_1,j_2,\ldots,j_m)}(s)\Big|_{s=\frac{1}{w}} = \frac{1}{w^{\delta(G)}}\,\tilde{N}_{(j_1,j_2,\ldots,j_m)}(w)$$

$$= \frac{n_\tau w^\tau + \ldots + n_1 w + n_0}{w^{\delta(G)}} \quad . \qquad (1.12)$$

Thus

$$G_{(j_1,j_2,\ldots,j_m)}(s) = (n_\tau \frac{1}{s^\tau} + \ldots + n_1 \frac{1}{s} + n_0)s^{\delta(G)} \; .$$

Now since G(s) is polynomial $\tau \le \delta(G)$, in fact from (1.11) it follows that

$$\delta(G_{(j_1,j_2,\ldots,j_m)}(s)) = \delta(G) \; .$$

Thus a high order minor of degree $\delta(G)$ exists, as required.

The condition in theorem (1.2) that G(s) be of full rank is important as the following example illustrates.

<u>(1.13):Example</u>:Consider

$$G(s) = \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix}. \qquad (1.14)$$

Then by (2.2.11) G(s) has no finite poles and since the Smith form of G(s) is

$$S(G) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

G(s) has no finite zeros.

To consider the point at infinity substitute $s = \frac{1}{w}$ giving

$$G(\tfrac{1}{w}) = \begin{bmatrix} 1 & \frac{1}{w} \\ \frac{1}{w} & \frac{1}{w^2} \end{bmatrix}$$

and,

$$G(\tfrac{1}{w}) = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -w & w^2 \end{bmatrix}^{-1} \qquad (1.15)$$

is a relatively right prime factorisation of $G(\frac{1}{w})$. Hence

$$\tilde{D}(w) = \begin{bmatrix} 1 & 0 \\ -w & w^2 \end{bmatrix} \qquad (1.16)$$

is a denominator of $G(\frac{1}{w})$ while,

$$\tilde{N}(w) = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix} \qquad (1.17)$$

is a numerator.

It is then clear from (1.16) that G(s) has an infinite pole of degree two and from (1.17) that G(s) has no infinite zeros.

Thus, although there is no high order minor of degree $\delta(G) = 2$, still G(s) has no infinite zeros.

## Section (3.2): Discussion of Work by Gantmacher and Verghese

In this section work by Gantmacher (1959) and Verghese et al (1979) will be discussed. These authors have considered the structure at infinity of certain rational matrices, some of Verghese's work being based on Gantmacher's results. The results obtained by Gantmacher and Verghese will be compared and contrasted with the results developed thus far in this thesis and new results will be established, based on their ideas. Although the discussion to be presented is perhaps more naturally inclined to the contents of the previous chapter it appears here because of its partial reliance on theorem (1.2).

In particular, Gantmacher, whose work is based on Kronecker's (1867) pencil theory, discusses linear polynomial matrices of the form

$$A+sB$$

where A and B are constant matrices. Such a matrix is called a "pencil of matrices". Pencils of matrices are said to be either regular or singular according to the following definition.

<u>(2.1):Definition</u>: An mxn pencil of matrices A+sB is called regular if m=n and $|A+sB| \neq 0$.

In all other cases (i.e. $m \neq n$ or m=n and $|A+sB| = 0$) the pencil is called singular.

Gantmacher defines the finite elementary divisors of a regular pencil of matrices in the same way as the finite elementary divisors were defined in section (1.2). In order to define the infinite elementary divisors the pencil must first be written in terms of the homogeneous parameters $\mu$ and s as $\mu A+sB$. Then

<u>(2.2):Definition</u>: The infinite elementary divisors of the pencil A+sB are the elementary divisors of the form $\mu^q$ of $\mu A+sB$.

In fact, the infinite elementary divisors of A+sB are not the same as the elementary divisors of the form $w^q$ of numerators of the matrix $A+\frac{1}{w}B$ as the following example shows.

<u>(2.3):Example</u>: Consider the pencil

$$A+sB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (2.4)$$

This pencil is regular since $|A+sB| \neq 0$. Introducing the parameter $\mu$ gives

$$\mu A+sB = \begin{bmatrix} \mu & \mu+s \\ 0 & \mu \end{bmatrix}$$

which has elementary divisor $\mu^2$. Hence A+sB has one infinite elementary divisor of degree 2.

Substituting $s = \frac{1}{w}$ in (2.4) gives

$$A+\frac{1}{w}B = \begin{bmatrix} 1 & \frac{w+1}{w} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & w+1 \\ 0 & w \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$$

which is a relatively right prime factorisation. Hence the numerator of $A+ \frac{1}{w}B$ has one elementary divisor w, i.e. according to definition (2.1.25) A+sB has one infinite zero of degree 1.

This result shows that the infinite elementary divisors of a matrix pencil do not correspond directly to the elementary divisors used to define its infinite zeros. Thus although Gantmacher shows that A+sB has no infinite elementary divisors if and only if $|B| \neq 0$ one would not expect this to yield a result concerning infinite zeros. The next result partially supports this by indicating that the condition $|B| \neq 0$ is a sufficient but not a necessary condition for A+sB to have no infinite zeros.

<u>(2.5):Theorem</u>: If $|B| \neq 0$ then A+sB has no infinite zeros.

<u>Proof</u>: Let

$$A+sB = A+ \frac{1}{w}B$$
$$= (wA+B)(wI)^{-1}.$$

This is a relatively right prime factorisation since $|B| \neq 0$. Also, the numerator has full rank at w=0 and hence A+sB has no infinite zeros.

This result is also a direct consequence of theorem (1.2) since if $|B| \neq 0$, $|A+sB|$ must have degree m. Clearly A+sB cannot possess a minor of any higher degree. Consequently

$$\delta(A+sB) = m$$
$$= \delta\ |A+sB|$$

and, by theorem (1.2), A+sB has no infinite zeros.

The fact that the converse of theorem (2.5) is not true is illustrated by the next example.

<u>(2.6):Example</u>: Consider

$$A+sB = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$$

This is a regular pencil of matrices since it is square and $|A+sB| \neq 0$. Note however that $|B|=0$. Now

$$A+\frac{1}{w}B = \begin{bmatrix} \frac{1}{w} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

is a relatively right prime factorisation and thus A+sB has no infinite zeros even though $|B|=0$. Note however that, according to Gantmacher, A+sB has one infinite elementary divisor.

Further connections with Gantmacher's work have been established by Verghese et al (1979). Verghese refers to the poles and zeros at infinity of the rational matrix $Q(s)$ as the poles and zeros at $s=0$ of $Q(s^{-1})$. This is of course consistent with definition (2.1.25). Verghese proves the following result.

<u>(2.7):Theorem</u>: Let

$$Q(s) = sK-L$$

be a matrix pencil. Use constant non-singular transformations to bring Q(s) to the form

$$Q_1(s) = s\begin{bmatrix} K_1 \\ 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{2.8}$$

where $K_1$ has full row rank. (This operation preserves the pole-zero structure of sK-L). Then the zero structure of sK-L at infinity is isomorphic to the structure of

$$\begin{bmatrix} K_1 - L_1 s \\ -L_2 \end{bmatrix}$$

at s=0.

The construction used in this theorem involves performing constant non-singular transformations on Q(s). Whilst it is clear that such transformations do not affect the finite zeros of Q(s) (since these operations are particular transformations of unimodular equivalence) it has not been established so far in this thesis that such transformations leave unchanged the infinite zeros of Q(s). Although intuitively plausible, this result will be formally established in the next chapter. It is easy to see that the infinite zeros as defined in theorem (2.7) are those defined in (2.1.25).

<u>(2.9):Theorem</u>: With the notation of theorem (2.7) the zeros at s=0 of

$$\begin{bmatrix} K_1 - L_1 s \\ -L_2 \end{bmatrix} \qquad (2.10)$$

are the zeros at w=0 of $Q(\frac{1}{w})$.

<u>Proof</u>: Substitute $s = \frac{1}{w}$ in (2.8) giving

$$Q_1(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w}K_1 - L_1 \\ -L_2 \end{bmatrix}$$

$$= \begin{bmatrix} wI & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} K_1 - wL_1 \\ -L_2 \end{bmatrix}. \qquad (2.11)$$

Clearly

$$\begin{bmatrix} wI & 0 & K_1-wL_1 \\ 0 & I & -L_2 \end{bmatrix}$$

has full row rank since $K_1$ has full row rank and thus (2.11) is a relatively left prime factorisation of $Q_1(\frac{1}{w})$. Hence

$$\begin{bmatrix} K_1-wL_1 \\ -L_2 \end{bmatrix} \tag{2.12}$$

is a numerator of $Q_1(\frac{1}{w})$ and, since $Q(\frac{1}{w})$ and $Q_1(\frac{1}{w})$ are related by a constant non-singular transformation, (2.12) is also a numerator of $Q(\frac{1}{w})$. Clearly (2.10) and (2.12) have the same set of zeros and the result follows immediately.

Gantmacher describes constant non-singular transformations by which any regular pencil of matrices A+sB may be brought to the form

$$\begin{bmatrix} sI-A_1 & 0 \\ 0 & I+sJ \end{bmatrix} \tag{2.13}$$

where the quasi-diagonal matrix I+sJ is made up of kxk blocks of the form

$$(I_k+sJ_k) = \begin{bmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{2.14}$$

By theorem (2.5) it is clear that $(sI-A_1)$ has no infinite elementary divisors and also no infinite zeros. Hence, since (2.13) was obtained from A+sB by constant non-singular transformations, the infinite elementary divisors and the infinite zeros of A+sB are simply those of I+sJ in (2.13).

Verghese notes without proof that a k-th order elementary divisor of a matrix pencil corresponds to a (k-1)th order infinite zero. This result is supported by examples (2.3) and (2.6) and a simple proof for the case of regular matrix pencils is now offered. This proof can be easily extended to the case of singular matrix pencils.

<u>(2.15):Theorem</u>: A k th order infinite elementary divisor of the regular matrix pencil A+sB corresponds to a (k-1)th order infinite zero.

<u>Proof</u>: Consider the kxk block matrix $(I_k+sJ_k)$ as given in (2.14).

Introducing the parameter μ gives

$$\mu I_k+sJ_k = \begin{bmatrix} \mu & s & 0 & \cdots & \cdots & 0 \\ 0 & \mu & s & \ddots & & \vdots \\ \vdots & 0 & \mu & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \mu & s \\ 0 & \cdots & \cdots & \cdots & 0 & \mu \end{bmatrix}$$

which has one elementary divisor of the form $\mu^k$. i.e. $I_k+sJ_k$ has one $k$th order infinite elementary divisor.

Now substitute $s = \frac{1}{w}$ in (2.14) giving

$$I_k + \frac{1}{w} J_k = \begin{bmatrix} 1 & \frac{1}{w} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \frac{1}{w} & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 & \frac{1}{w} \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w & 0 & \cdots & \cdots & 0 \\ 0 & w & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & w & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} w & 1 & 0 & \cdots & \cdots & 0 \\ 0 & w & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & w & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

which is a relatively left prime factorisation. The numerator has Smith form

$$\begin{bmatrix} I_{k-1} & 0_{k-1,1} \\ 0_{1,k-1} & w^{k-1} \end{bmatrix}$$

and hence it is clear that $(I_k+ \frac{1}{w}J_k)$ possesses one zero of degree k-1 at w=0, i.e. $I_k+sJ_k$ has one (k-1)th order infinite zero.

Hence each kxk block matrix $I_k+sJ_k$ in (2.13) possesses an infinite elementary divisor of degree k but only an infinite zero of degree k-1 and the result follows.

Verghese also proves an interesting result which relates the total number of poles, both finite and infinite, of a rational matrix G(s) to the total number of zeros. This result will be stated here for completeness. However, since the theorem involves the minimal indices of the left and right null spaces of G(s) these indices must first be defined. The following definition is due to Rosenbrock (1970 p.96).

<u>(2.16):Definition</u>: Let A+sB be a singular pencil of matrices. The equation

$$(A+sB)\ x(s)=0 \qquad (2.17)$$

admits polynomial solution vectors x(s). Among all the solution vectors there will be one, say $x_1(s)$, with lowest degree, say $\lambda_1$. A second solution vector, say $x_2(s)$ will be linearly dependent on $x_1(s)$ if there exist polynomials $\alpha(s)$, $\beta(s)$ such that

$$\alpha(s)\ x_1(s) + \beta(s)\ x_2(s) = 0.$$

If no such $\alpha(s)$, $\beta(s)$ exist $x_1(s)$ and $x_2(s)$ are linearly independent. Among all the solutions of (2.17) linearly independent of $x_1(s)$ there will be at least one, $x_2(s)$, which has the lowest degree, say $\lambda_2$. Proceeding in this way successive polynomial vectors $x_1(s)$, $x_2(s)$, $x_3(s)$, ... may be generated, each independent of those which preceed it until no more can be found. The set $\{\lambda_i\}$ is defined to be the set of MINIMAL INDICES OF THE RIGHT NULL SPACE of A+sB.

Similarly, the set of minimal indices of the left null space of A+sB may be found by considering the polynomial solution row vectors of the equation.

$$y(s)(A+sB)=0 \tag{2.18}$$

Verghese shows the relationship between the number of poles and zeros of a rational matrix in the following result.

<u>(2.19):Theorem</u>: Let $\delta_p(G)$ and $\delta_z(G)$ denote the total number of poles and zeros respectively, both finite and infinite, of the rational matrix G(s) and let $\alpha(G)$ denote the sum of the minimal indices of the left and right null spaces of G(s). Then

$$\delta_p(G) = \delta_z(G) + \alpha(G). \tag{2.20}$$

Note that if G(s) is square and non-singular, equations (2.17) and (2.18) have no solutions and thus G(s) has no minimal indices. Hence, in this case,

$$\alpha(G) = 0$$

and (2.20) gives

$$\delta_p(G) = \delta_z(G).$$

This result is consistent with corollary (2.3.20) which was proved by a completely different approach.

Section (3.3): Interpretations of Minimal Bases.

Forney (1975) viewed a basis for a rational vector space over the field of rational functions as being a rational matrix whose rows or columns are linearly independent. From such a basis a polynomial basis (consisting solely of polynomial vectors) may always be constructed. Let P(s) be an $m \times \ell$ polynomial matrix. Assume that $m \leq \ell$ although this in no way restricts what is to be said, it being adopted in this and the next section merely for the purpose of the exposition. Note that analogous conclusions may be drawn if $m > \ell$ if the terms "rows" and "columns" are interchanged, while if $m = \ell$ either of these terms may be used. Suppose that the normal rank of P(s), denoted $\rho(P)$, is m and the degree of the i th row of P(s) is $\delta_i$ $(i = 1,2,\ldots,m)$. The high order coefficient matrix of P(s), denoted $[P]_h$, is that matrix whose i, j th element is the coefficient of $s^{\delta_i}$ in the i, j th element of P(s). Using this terminology a minimal basis (Forney (ibid)) is defined as follows:

(3.1): Definition: The rows of P(s) are (or simply, P(s) is) said to form a minimal basis if

(i) P(s) has full rank for all finite $s \in \mathbb{C}$

and (ii) $[P]_h$ has full rank.

If $[P]_h$ has full row rank then P(s) is said to be ROW PROPER. Thus condition (ii) of definition (3.1) requires that a minimal basis must be row proper.

Anderson and Bitmead (1978) have given an alternative definition of infinite zeros. With this definition they have shown that column or row properness (Wolovich 1974) of a polynomial matrix can be interpreted as an absence of zeros at infinity. In this section it will be shown that column or row properness is a sufficient but not a necessary condition for the absence of infinite zeros as defined in (2.1.25). However, the importance of column or row properness will become clear in the next section when this property will be shown to have a valuable structural implication for factorisations of rational matrices.

The following result can be established as a simple corollary to theorem (1.2). However, the direct proof given here is relevant to a later result in this section.

<u>(3.2):Theorem</u>: If the $m \times \ell$ polynomial matrix $P(s)$ forms a minimal basis then it possesses no finite poles and no finite or infinite zeros.

<u>Proof</u>: Since $P(s)$ is polynomial it has no finite poles. Also, from (i) of definition (4.1) it follows that the Smith form of $P(s)$ is $(I_m, O_{\ell-m})$ and hence $P(s)$ has no finite zeros.

From (ii) of definition (4.2), $[P]_h$ has full rank. Let

$$\Lambda(s) = \mathrm{diag}(s^{\delta_1}, \ldots, s^{\delta_m}) \qquad (3.3)$$

Then

$$P(\tfrac{1}{w}) = \Lambda(\tfrac{1}{w})\tilde{P}(w)$$
$$= \Lambda^{-1}(w)\tilde{P}(w) \qquad (3.4)$$

is a polynomial factorisation of $P(\frac{1}{w})$ where

$$\tilde{P}(0) = \left[P\right]_h . \qquad (3.5)$$

Consider the matrix

$$(\Lambda(w), \tilde{P}(w)) . \qquad (3.6)$$

For w=0

$$(\Lambda(0), \tilde{P}(0)) \equiv (0_{m,m}, \left[P\right]_h)$$

which has full row rank since $\left[P\right]_h$ has rank m. For any finite $w(\neq 0) \in \mathbb{C}$ the matrix (3.6) certainly has full rank because of the form of $\Lambda(w)$. Hence $(\Lambda(w), \tilde{P}(w))$ has full rank for all finite $w \in \mathbb{C}$ and consequently (3.4) is a relatively left prime polynomial factorisation of $P(\frac{1}{w})$.

It thus follows that $\Lambda(w)$ is a denominator and $\tilde{P}(w)$ is a numerator of $P(\frac{1}{w})$. In particular therefore the zeros of $P(\frac{1}{w})$ at w=0 are precisely the zeros at w=0 of $\tilde{P}(w)$. However

$$\text{RANK } \tilde{P}(0) = \text{RANK } \left[P\right]_h = m$$

and so P(w) has no zeros at w=0. Hence, from definition (3.1), P(s) has no infinite zeros as required.

Note that condition (i) from definition (3.1) implies that a minimal basis P(s) has no finite zeroswhile (ii) implies that P(s) has no infinite zeros.In fact (i) is both a necessary and sufficient condition for the absence of finite zeros whereas (ii) is merely a sufficient but not a necessary condition for the absence of infinite zeros as the next example shows.

(3.7): Example: Let

$$P_1(s) = \begin{bmatrix} s & 0 & 1 \\ s & 1 & 1 \end{bmatrix} \tag{3.8}$$

then $\rho(P_1) = 2$. Thus $P_1(s)$ is a polynomial basis for a certain rational vector space.

In fact $P_1(s)$ has full rank for all finite $s \varepsilon \mathbb{C}$ but $P_1(s)$ is not minimal since

$$\left[P_1\right]_h = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which is clearly not of rank 2.

However, consider the infinite zeros of $P_1(s)$. Now

$$P_1(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w} & 0 & 1 \\ \frac{1}{w} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \end{bmatrix}$$

which is a relatively left prime factorisation. Clearly the numerator has full rank at $w=0$ and consequently $P_1(s)$ has no infinite zeros.

This example demonstrates that when infinite zeros are defined as in (1.2.25) row properness is not an exact characterisation of polynomial matrices possessing no infinite zeros. In a recent paper Anderson and Bitmead (1978) give an essentially different definition of the term "infinite zeros" under which row properness is an exact characterisation of such matrices. Attractive though such a definition may appear from this point of view it is highly unsatisfactory from another. For under their definition the matrix $P_1(s)$ of example (3.7) and

$$\begin{bmatrix} s & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.9}$$

do not have the same infinite zeros. However

$$P_1(s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

so that $P_1(s)$ and (3.9) are only a simple constant transformation away from each other. Thus the infinite zeros as defined by Anderson and Bitmead (ibid) are not invariant under the simplest of transformations. This difficulty arises because the construction used by Anderson and Bitmead introduces "infinite zeros" over and above those that were defined in (2.1.25). It is these additional quantities which give rise to the situation just described and as a consequence represent dynamically uninteresting properties.

It is seen from example (3.7) that row properness is a sufficient condition for the exclusion of infinite zeros in the sense of definition (2.1.25), but not a necessary one. Thus from the point of view of the existence of infinite zeros the concept of row properness does not assume great importance. Despite this however the concept does possess a very interesting implication for relatively prime factorisations of rational matrices as will be seen in the next section.

## Section (3.4): Minimal Factorisations

In the last section it was shown that a matrix which has no infinite zeros is not necessarily row or column proper and consequently, these concepts are of no great importance when considering the existence of infinite zeros. However, the concept of row or column properness does have a valuable application to the theory of matrix factorisations.

Specifically this section will distinguish between the usual relatively prime factorisations and what are termed minimal factorisations of a given rational matrix G(s). Relatively prime factorisations by definition display the finite poles and zeros of G(s) and minimal factorisations are demonstrated to possess an important structural property in their ability to display both the finite and infinite poles and zeros of G(s). This property will be utilised in the final chapter in the investigation of the effect of output feedback on the poles and zeros of the transfer function matrix.

The following theorem presents a fundamental structural property of row and column proper factorisations which forms one of the central results of this thesis.

(4.1):Theorem: Let G(s) be an $m \times \ell$ rational matrix and let

$$G(s) = D^{-1}(s)N(s) \qquad (4.2)$$

be a polynomial factorisation of G(s) in which the matrix

$$(D(s),\ N(s)) \qquad (4.3)$$

forms a minimal basis. Let the ith row degree of (4.3) be denoted by $\delta_i$ $(i=1,2,...,m)$ and define

$$\Lambda(s) = \mathrm{diag}(s^{\delta_1}, s^{\delta_2}, \ldots, s^{\delta_m}) \tag{4.4}$$

Then

(i) the finite poles of $G(s)$ are the finite zeros of $D(s)$ and the infinite poles of $G(s)$ are the zeros at $w=0$ of the polynomial matrix

$$\Lambda(w)D(\tfrac{1}{w}) \tag{4.5}$$

(ii) the finite zeros of $G(s)$ are the finite zeros of $N(s)$ and the infinite zeros of $G(s)$ are the zeros at $w=0$ of the polynomial matrix

$$\Lambda(w)N(\tfrac{1}{w}). \tag{4.6}$$

<u>Proof</u>: Since $(D(s),N(s))$ is a minimal basis the matrices $D(s)$ and $N(s)$ are relatively left prime by definition (3.1). Thus (4.2) is a relatively left prime factorisation of $G(s)$ and so the statements concerning the finite poles and zeros follows immediately from their definition.

Now consider the infinite case. From definition (3.1) the high order coefficient matrix of (4.3), denoted $[D,N]_h$, has full row rank. Now

$$\begin{aligned}(D(\tfrac{1}{w}),N(\tfrac{1}{w})) &= \Lambda(\tfrac{1}{w})(\tilde{D}(w),\tilde{N}(w)) \\ &= (\Lambda(w))^{-1}(\tilde{D}(w),\tilde{N}(w)) \end{aligned} \tag{4.7}$$

is a polynomial factorisation of $(D(\frac{1}{w}),N(\frac{1}{w}))$ where

$$(\tilde{D}(0),\tilde{N}(0)) = [D,N]_h .$$

Hence the matrix

$$(\Lambda(w),\tilde{D}(w),\tilde{N}(w)) \tag{4.8}$$

has full row rank at $w=0$ since $[D,N]_h$ has full row rank. For any finite $w\in\mathbb{C}$, where $w\neq 0$, (4.8) has full row rank because of the form of $\Lambda(w)$. Thus (4.7) is a relatively left prime factorisation of $(D(\frac{1}{w}),N(\frac{1}{w}))$.

Since $(D(\frac{1}{w}),N(\frac{1}{w}))$ was formed by performing the bilinear transformation $s = \frac{1}{w}$ on $(D(s),N(s))$ the finite zeros of $(D(\frac{1}{w}),N(\frac{1}{w}))$ will correspond to the non-zero finite and infinite zeros of $(D(s),N(s))$ in a one-to-one manner. However, by theorem (3.2) a minimal basis such as $(D(s), N(s))$ has no zeros, finite or infinite and consequently any numerator of $(D(\frac{1}{w}),N(\frac{1}{w}))$ such as $(\tilde{D}(w),\tilde{N}(w))$ has no finite zeros. Thus $(\tilde{D}(w),\tilde{N}(w))$ has full row rank for all finite w. i.e. $\tilde{D}(w)$ and $\tilde{N}(w)$ are relatively left prime.

Now from (4.7)

$$\tilde{D}(w) = \Lambda(w)D(\tfrac{1}{w}),\quad \tilde{N}(w) = \Lambda(w)N(\tfrac{1}{w}) \tag{4.9}$$

and so

$$\begin{aligned}\tilde{D}^{-1}(w)\tilde{N}(w) &= D^{-1}(\tfrac{1}{w})\Lambda^{-1}(w)\Lambda(w)N(\tfrac{1}{w})\\ &= D^{-1}(\tfrac{1}{w})N(\tfrac{1}{w}).\end{aligned}$$

Hence, by (4.2)

$$G(\tfrac{1}{w}) = \tilde{D}^{-1}(w)\tilde{N}(w).$$

But $\tilde{D}(w)$ and $\tilde{N}(w)$ are relatively left prime polynomial matrices and so, by definition (2.1.25), the infinite poles of $G(s)$ are the zeros at $w=0$ of $\tilde{D}(w)$ and the infinite zeros of $G(s)$ are the zeros at $w=0$ of $\tilde{N}(w)$. By virtue of (4.9) the theorem follows as required.

It is thus seen that not only do the factorisations described in the theorem display the finite poles and zeros of the underlying rational matrix (as indeed does any relatively prime factorisation) but they additionally display the infinite poles and zeros in a particularly simple way.

This structural property is extremely useful in the output feedback investigation to be carried out in chapter 5. It therefore seems appropriate to distinguish between such factorisations of a given rational matrix G(s) and the usual relatively prime factorisations in the manner proposed by Forney (1975). Accordingly factorisations of the type described in theorem (4.1) will be termed MINIMAL FACTORISATIONS.

Of course one technical advantage of a minimal factorisation is that it obviates the need to construct a second factorisation of the rational matrix G(s) when the point at infinity is to be considered. Thus all the information presented by the relatively prime factorisations of both G(s) and $G(\frac{1}{w})$ is presented concisely within a single (minimal) factorisation of G(s).

The following example illustrates the result of this theorem.

<u>4.10:Example</u>: Let

$$G(s) = \begin{bmatrix} \frac{1}{(s-1)(s-2)} & \frac{s}{s-1} \\ \frac{-s}{s-2} & 1-2s \end{bmatrix} \qquad (4.11)$$

As a first attempt at finding a relatively left prime factorisation of G(s) put

$$G(s) = \begin{bmatrix} (s-1)(s-2) & 0 \\ 0 & (s-2) \end{bmatrix}^{-1} \begin{bmatrix} 1 & s(s-2) \\ -s & (1-2s)(s-2) \end{bmatrix}$$

$$= D_1^{-1}(s)N_1(s).$$

But $(D_1(s),N_1(s))$ does not have full row rank so this polynomial factorisation of G(s) is not relatively left prime i.e. the order of $(D_1(s),N_1(s))$ must be reduced following the method described by Rosenbrock (1970 p.60). Adding twice row 1 to row 2 of $(D_1(s),N_1(s))$ gives

$$(D_2(s),N_2(s)) = \begin{bmatrix} (s-1)(s-2) & 0 & 1 & s(s-2) \\ 2(s-1)(s-2) & s-2 & 2-s & s-2 \end{bmatrix}$$

where all the elements in the second row contain the factor (s-2).

Dividing row 2 by this factor gives

$$(D_3(s),N_3(s)) = \begin{bmatrix} (s-1)(s-2) & 0 & 1 & s(s-2) \\ 2(s-1) & 1 & -1 & 1 \end{bmatrix}$$

which has full row rank for all finite s. Thus

$$D_3^{-1}(s)N_3(s) = \begin{bmatrix} (s-1)(s-2) & 0 \\ 2(s-1) & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s(s-2) \\ -1 & 1 \end{bmatrix} \qquad (4.12)$$

is a relatively left prime factorisation of G(s). Now the high order coefficient matrix of $(D_3(s),N_3(s))$ is

$$\left[D_3,N_3\right]_h = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which has full row rank. Hence (4.12) is also a minimal factorisation of G(s). If (4.12) had not been a minimal factorisation it would have been necessary to follow Forney's (1975) algorithm to reduce $(D_3(s),N_3(s))$ to a minimal basis and thus form a minimal factorisation of G(s).

From (4.12)

$$N_3(s) = \begin{bmatrix} 1 & s(s-2) \\ -1 & 1 \end{bmatrix}$$

is a numerator of G(s) and this matrix has Smith form

$$S(N_3) = \begin{bmatrix} 1 & 0 \\ 0 & (s-1)^2 \end{bmatrix} .$$

i.e. G(s) has a finite zero of degree two at s=1.

Similarly

$$D_3(s) = \begin{bmatrix} (s-1)(s-2) & 0 \\ 2(s-1) & 1 \end{bmatrix}$$

which has Smith form

$$S(D3) = \begin{bmatrix} 1 & 0 \\ 0 & (s-1)(s-2) \end{bmatrix}$$

is a denominator of G(s) and hence G(s) has one finite pole of degree one at s=1 and one finite pole of degree one at s=2.

Now, following theorem (4.1)

$$\Lambda(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} .$$

Thus a numerator of $G(\frac{1}{w})$ is given by

$$\Lambda(w)N(\tfrac{1}{w}) = \begin{bmatrix} w^2 & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{w}(\frac{1}{w}-2) \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w^2 & (1-2w) \\ -w & w \end{bmatrix}$$

which has Smith form

$$\begin{bmatrix} 1 & 0 \\ 0 & w(w^2-2w+1) \end{bmatrix}$$

Consequently, by theorem (4.1), G(s) has one infinite zero of degree 1.

A denominator of $G(\frac{1}{w})$ is given by

$$\Lambda(w)D(\frac{1}{w}) = \begin{bmatrix} w^2 & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} (\frac{1}{w}-1)(\frac{1}{w}-2) & 0 \\ 2(\frac{1}{w}-1) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1-w)(1-2w) & 0 \\ 2(1-w) & w \end{bmatrix}$$

which has Smith form

$$\begin{bmatrix} 1 & 0 \\ 0 & w(1-w)(1-2w) \end{bmatrix} .$$

Thus G(s) has one infinite pole of degree one.

Of course the infinite poles and zeros of G(s) could have been found directly by considering any relatively prime polynomial factorisation of

$$G(\frac{1}{w}) = \begin{bmatrix} \frac{w^2}{(1-w)(1-2w)} & \frac{1}{1-w} \\ \frac{-1}{1-2w} & \frac{w-2}{w} \end{bmatrix} .$$

In a first attempt to find a relatively left prime factorisation of $G(\frac{1}{w})$ try

$$G(\frac{1}{w}) = \begin{bmatrix} (1-w)(1-2w) & 0 \\ 0 & w(1-2w) \end{bmatrix}^{-1} \begin{bmatrix} w^2 & (1-2w) \\ -w & (w-2)(1-2w) \end{bmatrix} .$$

$$= D_4^{-1}(s)N_4(s).$$

However, this polynomial factorisation is not relatively left prime and consequently the method described by Rosenbrock must be utilised once again to reduce the order of the matrix $(D_4(s),N(s))$. In this case this can be achieved by adding twice the first row to the second row and then dividing the second row by the factor (1-2w) to get

$$(D_5(s), N_5(s)) = \begin{bmatrix} (1-w)(1-2w) & 0 & w^2 & 1-2w \\ 2(1-w) & w & -w & w \end{bmatrix}$$

which has full row rank for all finite w.

i.e. $G(\frac{1}{w}) = D_5^{-1}(w)N_5(w)$

$$= \begin{bmatrix} (1-w)(1-2w) & 0 \\ 2(1-w) & w \end{bmatrix}^{-1} \begin{bmatrix} w^2 & 1-2w \\ -w & w \end{bmatrix}$$

is a relatively left prime factorisation of $G(\frac{1}{w})$.

Consequently, the infinite zeros of G(s) are the zeros at w=0 of the Smith form of $N_5(w)$ and the infinite poles of G(s) are the zeros at w=0 of $D_5(w)$. As would be expected this method yields the same results as the method described in theorem (4.1).

In many cases, as in the above example, the method described in theorem (4.1) for finding the infinite poles and zeros of a rational matrix will be computationally simpler than finding and examining a relatively prime factorisation of $G(\frac{1}{w})$. This is because the most computationally difficult step involved in the calculation of the finite and infinite poles and zeros of a rational matrix, namely the finding of a relatively prime factorisation, is encountered only once using the method of theorem (4.1).

Irrespective of whether or not the results described in this section offer any computational advantage in a particular example their main importance is from a theoretical point of view as will be seen subsequently.

Chapter 4. Applications in Linear Systems.

## Section (4.1): Infinite Decoupling Zeros and Transfer Function Zeros

The theory concerning the behaviour of linear multivariable systems at finite frequencies has been widely researched and is well understood. The relevant results for this thesis were summarised in chapter 1. In this chapter the results developed in the two previous chapters will be applied to system matrices and transfer function matrices and thus the behaviour of the system at infinite frequencies in particular will be investigated.

Accordingly consider the polynomial system matrix

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{1.1}$$

and its associated transfer function matrix

$$G(s) = V(s)T^{-1}(s)U(s) + W(s). \tag{1.2}$$

The finite decoupling zeros of P(s) were defined in section (1.2). In order to define the infinite decoupling zeros the system matrix P(s) must be trivially expanded to form the normalised system matrix

$$P_N(s) = \left[\begin{array}{ccc:c} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & -I & 0 \\ 0 & I & 0 & -I \\ \hdashline 0 & 0 & I & 0 \end{array}\right]. \tag{1.3}$$

Clearly P(s) and $P_N(s)$ give rise to the same transfer function matrix. Verghese (1978) defines the finite and infinite decoupling zeros of the system P(s) as follows

(1.4) :Definition: The finite input-decoupling zeros of the system $P(s)$ are the finite zeros of

$$(T(s) \quad U(s)) \tag{1.5}$$

and the infinite input-decoupling zeros are the zeros at $w=0$ of

$$\begin{bmatrix} T(\frac{1}{w}) & U(\frac{1}{w}) & 0 \\ -V(\frac{1}{w}) & W(\frac{1}{w}) & -I \end{bmatrix}. \tag{1.6}$$

The finite output-decoupling zeros of $P(s)$ are the finite zeros of

$$\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix} \tag{1.7}$$

and the infinite output-decoupling zeros are the zeros at $w=0$ of

$$\begin{bmatrix} T(\frac{1}{w}) & U(\frac{1}{w}) \\ -V(\frac{1}{w}) & W(\frac{1}{w}) \\ 0 & I \end{bmatrix}. \tag{1.8}$$

Verghese does not specifically define the infinite input-output-decoupling zeros of $P(s)$. However, he implies that, in line with the finite case, the infinite input-output-decoupling zeros of $P(s)$ are simply the infinite input-decoupling zeros which are at the same time infinite output-decoupling zeros of $P(s)$.

In fact both the finite and infinite decoupling zeros as defined in (1.4) are just the finite and infinite zeros of the relevant part of the normalised system matrix as the next result shows.

(1.9) Theorem: The finite and infinite input-decoupling zeros of the system P(s) are the finite and infinite zeros of

$$\left[\begin{array}{ccc:c} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & -I & 0 \\ 0 & I & 0 & -I \end{array}\right] \tag{1.10}$$

and the finite and infinite output-decoupling zeros are the finite and infinite zeros of

$$\left[\begin{array}{ccc} T(s) & U(s) & 0 \\ -V(s) & W(s) & -I \\ 0 & I & 0 \\ \hdashline 0 & 0 & I \end{array}\right]. \tag{1.11}$$

Proof: By simple column operations (1.10) can be brought to the form

$$\left[\begin{array}{ccc:c} T(s) & U(s) & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{array}\right]$$

which clearly has the same finite zeros as (1.5). Hence (1.5) and (1.10) have the same set of finite zeros.

From definition (1.4) the infinite input-decoupling zeros of P(s) are the zeros of w=0 of (1.6). Let a relatively left prime factorisation of (1.6) be

$$\begin{bmatrix} T(\frac{1}{w}) & U(\frac{1}{w}) & 0 \\ -V(\frac{1}{w}) & W(\frac{1}{w}) & -I \end{bmatrix} = \tilde{D}^{-1}(w)\,\tilde{N}(w) \tag{1.12}$$

Now the infinite zeros of (1.10) are the zeros at $w=0$ of

$$\begin{bmatrix} T(\frac{1}{w}) & U(\frac{1}{w}) & 0 & 0 \\ -V(\frac{1}{w}) & W(\frac{1}{w}) & -I & 0 \\ 0 & I & 0 & -I \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}(w) & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}(w) & 0 \\ (0\ ,\ I\ ,\ 0) & -I \end{bmatrix}. \qquad (1.13)$$

Since, from the relatively left prime factorisation (1.12) the matrix $(\tilde{D}(w)\ ,\ \tilde{N}(w))$ has full row rank the matrix

$$\begin{bmatrix} \tilde{D}(w) & 0 & \tilde{N}(w) & 0 \\ 0 & I & (0,\ I,\ 0) & -I \end{bmatrix} \qquad (1.14)$$

also has full row rank and hence (1.13) is also a relatively left prime factorisation.

But the numerator of (1.13) can be reduced by constant column operations to

$$\begin{bmatrix} \tilde{N}(w) & 0 \\ 0 & I \end{bmatrix}$$

which is a trivial expansion of $\tilde{N}(w)$. Hence (1.12) and (1.13) have the same set of finite zeros i.e. (1.6) and (1.10) have the same infinite zeros.

The result concerning the output-decoupling zeros can be proved in a similar manner.

In the discussion that follows the finite and infinite decoupling zeros will be referred to via definition (1.4) although of course the interpretation involving the normalised system matrix could equally well be used as theorem (1.9) has shown.

Rosenbrock (1970 and 1974b) has established the relationship between the finite poles and zeros of the transfer function matrix G(s) and the finite decoupling zeros of P(s) as follows. The finite POLES OF THE SYSTEM P(s) are defined as the zeros of T(s) and denoted by $\{n_i\}$. Then

$$\{n_i\} = \{g_i, \beta_i, \gamma_i\} - \{\delta_i\} \tag{1.15}$$

where $\{g_i\}$ denotes the poles of G(s) and $\{\beta_i\}$, $\{\gamma_i\}$ and $\{\delta_i\}$ denote the input, output and input-output decoupling zeros of P(s) respectively. The set of ZEROS OF THE SYSTEM P(s) is the set $\{\alpha_i\}$ given by

$$\{\alpha_i\} \equiv \{t_i, \beta_i, \gamma_i\} - \{\delta_i\} \tag{1.16}$$

where $\{t_i\}$ denotes the zeros of G(s). As was described in section (1.2) Rosenbrock (1974b) established the following characterisation of system zeros.

<u>(1.17) Lemma</u>: The system zeros $\{\alpha_i\}$ are the zeros of the greatest common divisor of the bordered minors of P(s) of order p, i.e. of the form

$$P\begin{pmatrix} i_1, \ldots i_p \\ j_1, \ldots j_p \end{pmatrix} \tag{1.18}$$

where p is the greatest integer for which P(s) possesses as non-zero minor of the form (1.18).

These results were further discussed by Pugh (1977). Recently Verghese (1978) and Ferreira (1980) have shown that analogous results hold for the normalised system when either finite or infinite frequencies are considered.

Specifically Verghese (ibid p124) shows that for any $s_o \varepsilon \mathbb{C}$, including $s = \infty$,

$$\begin{bmatrix}\text{No. of zeros} \\ \text{of } T_N(s) \text{ at } s_o\end{bmatrix} = \begin{bmatrix}\text{No. of poles} \\ \text{of } G(s) \text{ at } s_o\end{bmatrix} + \begin{bmatrix}\text{No. of i.d.} \\ \text{zeros of} \\ P(s) \text{ at } s_o\end{bmatrix} + \begin{bmatrix}\text{No. of o.d.} \\ \text{zeros of} \\ P(s) \text{ at } s_o\end{bmatrix} - \begin{bmatrix}\text{No. of i.o.d.} \\ \text{zeros of } P(s) \\ \text{at } s_o\end{bmatrix} \qquad (1.19)$$

where

$$T_N(s) = \begin{bmatrix} T(s) & U(s) & 0 \\ -V(s) & W(s) & -I \\ 0 & I & 0 \end{bmatrix}$$

and the infinite decoupling zeros are defined as in definition (1.4). Hence equation (1.15) can be extended to include both the finite and infinite poles and zeros.

Ferreira (ibid) extends equation (1.16) to include the zeros at infinity. It is not immediately clear what is meant by "infinite system zeros". Ferreira defines them thus:

<u>(1.20) :Definition</u>: Consider the normalised system matrix $P_N(s)$ (as in equation (1.3)) and let the rank of $P_N(s)$ be $r+m+\ell+q$. Now let

$$P_N(\tfrac{1}{w}) = \tilde{N}(w)\tilde{D}^{-1}(w) \qquad (1.21)$$

be a relatively right prime factorisation of $P_N(\frac{1}{w})$. Clearly, like $P_N(\frac{1}{w})$, $\tilde{N}(w)$ has dimensions $(r+2\ell+m) \times (r+\ell+2m)$

and rank $r+m+\ell+q$. Now consider all the $(r+\ell+m+q)$-order minors of $\tilde{N}(w)$ which contain the first $r+\ell+m$ rows and columns. Let $d(w)$ be the greatest common divisor of all these minors. Then the number of infinite zeros of the system $P(s)$ (and also of $P_N(s)$) is defined as the number of zeros at $w=0$ of $d(w)$.

Using definition (1.20) Ferreira concludes that

$$\begin{bmatrix}\text{No.of infinite}\\ \text{system zeros}\end{bmatrix} = \begin{bmatrix}\text{No.of infinite}\\ \text{transfer}\\ \text{function zeros}\end{bmatrix} + \begin{bmatrix}\text{No.of infinite}\\ \text{i.d. zeros}\end{bmatrix} + \begin{bmatrix}\text{No.of infinite}\\ \text{o.d. zeros}\end{bmatrix} - \begin{bmatrix}\text{No.of infinite}\\ \text{i.o.d. zeros}\end{bmatrix} \quad (1.22)$$

and hence equation (1.16) can be extended to include the infinite zeros.

Neither Verghese nor Ferreira considers the multiplicity and degree of infinite zeros as defined in (2.1.25) and thus neither author finds it necessary to extend equations (1.19) and (1.22) to refer to the actual sets of zeros involved rather than simply considering the number of zeros. This extension can in fact be made so that

$$\begin{Bmatrix}\text{Zeros of } T_N(s)\\ \text{at } s_o\end{Bmatrix} = \begin{Bmatrix}\text{Poles of } G(s)\\ \text{at } s_o\end{Bmatrix} + \begin{Bmatrix}\text{I.d. zeros}\\ \text{of } P(s)\\ \text{at } s_o\end{Bmatrix} + \begin{Bmatrix}\text{O.d. zeros of}\\ P(s) \text{ at } s_o\end{Bmatrix} - \begin{Bmatrix}\text{I.o.d. zeros}\\ \text{of } P(s) \text{ at}\\ s_o\end{Bmatrix} \quad (1.23(a))$$

and

$$\left\{\begin{matrix}\text{System zeros}\\ \text{at } s_o\end{matrix}\right\} = \left\{\begin{matrix}\text{Zeros of}\\ G(s) \text{ at}\\ s_o\end{matrix}\right\} + \left\{\begin{matrix}\text{I.d. zeros}\\ \text{of } P(s) \text{ at}\\ s_o\end{matrix}\right\}$$

$$+ \left\{\begin{matrix}\text{O.d. zeros}\\ \text{of } P(s) \text{ at}\\ s_o\end{matrix}\right\} - \left\{\begin{matrix}\text{I.o.d. zeros}\\ \text{of } P(s) \text{ at}\\ s_o\end{matrix}\right\} \quad (1.23(b))$$

where $s_o \varepsilon \bar{\mathbb{C}}$. i.e. The point $s = \infty$ is included.

At first sight the definition of infinite decoupling zeros stated above (1.4) seems unnecessarily complicated and it is not obvious why. The infinite input-decoupling zeros of P(s), for example, are not simply the zeros at $w=0$ of $(T(\frac{1}{w}),\ U(\frac{1}{w}))$. However the results (1.19), (1.22), (1.23(a)) and (1.23(b)) form a natural extension of well-known results concerning finite system poles and zeros and hence they justify the use of definition (1.4).

A common representation of a linear multivariable system is the state-space realisation. Indeed any polynomial system matrix P(s) such as (1.1) may be transformed into the state-space form

$$P_1(s) = \left[\begin{array}{c:c} sI-A & B \\ \hdashline -C & D(s)\end{array}\right] \quad (1.24)$$

and Rosenbrock (1970) and Wolovich (1974) both describe algorithms for effecting this transformation. $P_1(s)$ has the same transfer function matrix and the same finite decoupling zeros as P(s) and thus the behaviour of P(s) at finite frequencies at least can be deduced from an analysis of the much simpler system matrix $P_1(s)$. However, when infinite frequency behaviour is considered the state-space form is no longer adequate and thus $P_1(s)$ has to be replaced by a system matrix in the GENERALISED STATE-SPACE FORM given by

$$P_2(s) = \left[\begin{array}{cc:c} sE & -A & B \\ \hdashline & -C & D(s) \end{array}\right]$$

where $|s\bar{E} - A| \neq 0$ although E may be singular. $P_2(s)$ may have infinite decoupling zeros whereas $P_1(s)$ does not as will be seen after the next result which specifically shows that if E is non-singular $P_2(s)$ has no infinite decoupling zeros.

<u>(1.25) : Theorem</u>: If $|E| \neq 0$ then the system matrix in the generalised state-space form has no infinite decoupling zeros.

<u>Proof</u>: The infinite input-decoupling zeros of $P_2(s)$ are the zeros at w=0 of

$$\begin{bmatrix} \frac{1}{w}E-A & B & 0 \\ -C & D(\frac{1}{w}) & -I \end{bmatrix} \tag{1.26}$$

$$= \begin{bmatrix} E-wA & B\tilde{D}(D) & 0 \\ -Cw & \tilde{N}(D) & -I \end{bmatrix} \begin{bmatrix} wI & 0 & 0 \\ 0 & \tilde{D}(D) & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \tag{1.27}$$

where

$$D(\tfrac{1}{w}) = \tilde{N}(D)\tilde{D}^{-1}(D)$$

is a relatively right prime factorisation of $D(\frac{1}{w})$. In the matrix

$$\begin{bmatrix} E-wA & B\tilde{D}(D) & 0 \\ -Cw & \tilde{N}(D) & -I \\ wI & 0 & 0 \\ 0 & \tilde{D}(D) & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (1.28)$$

the first block column has full column rank since $|E| \neq 0$ and the second block column has full column rank since $\tilde{N}(D)\tilde{D}^{-1}(D)$ is a relatively right prime factorisation of $D(\frac{1}{w})$. Consequently (1.28) has full column rank and (1.27) is a relatively right prime factorisation. But the numerator of (1.27) has full row rank at w=0 and hence $P_2(s)$ has no infinite input-decoupling zeros.

The result for the output-decoupling zeros can be proved similarly.

<u>(1.29) :Corollory</u>: The system matrix in the state-space form (1.24) has no infinite decoupling zeros.

<u>Proof</u>: This follows immediately since the state-space form is a special case of (1.24) with E = I.

Clearly therefore if a system matrix is transformed into the state-space form the result will have no infinite decoupling zeros and all the information relating to the infinite decoupling zeros of the original system will be lost. Consequently, when infinite frequency behaviour is being considered, realisation of the system matrix in the state-space form is not satisfactory and if a linearised system matrix is required then a more appropriate form is the generalised state-space system matrix .

The main operations used to generate state-space forms of system matrices are transformations of strict system equivalence and more generally extended strict system equivalence which in turn involve operations of unimodular equivalence and extended unimodular equivalence respectively. In fact neither of these operations in general preserves the infinite zero structure of a polynomial matrix for consider the following simple example.

<u>(1.30) :Example</u>: Let

$$P(s) = \begin{bmatrix} s & s^2 \\ 0 & 1 \end{bmatrix}. \tag{1.31}$$

This matrix can be transformed by unimodular equivalence to $P_1(s)$ where

$$P_1(s) = \begin{bmatrix} s & s^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s^2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & s^3+s^2 \\ 0 & 1 \end{bmatrix} . \tag{1.32}$$

Now

$$P(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w} & \frac{1}{w}2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} w & 1 \\ 0 & 1 \end{bmatrix}$$

which is a relatively left prime factorisation and hence $P(s)$ has one infinite zero of degree one. However

$$P_1(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w} & \frac{1}{w^3} + \frac{1}{w^2} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} w^2 & 1+w \\ 0 & 1 \end{bmatrix}$$

which is also a relatively left prime factorisation from which it can be seen that $P_1(s)$ has one infinite zero of degree two. Therefore, although $P(s)$ and $P_1(s)$ are unimodular equivalent they do not have the same infinite zero structure.

This example involves one of the simplest transformations of unimodular equivalence. Postmultiplying $P(s)$ by

$$\begin{bmatrix} 1 & s^2 \\ 0 & 1 \end{bmatrix}$$

has the effect of adding $s^2$ multiplied by the first column of $P(s)$ to the second column of $P(s)$. The example shows that even such a basic operation of unimodular equivalence has the effect of introducing an additional infinite pole and zero.

Clearly then the polynomial matrix operations of unimodular equivalence and extended unimodular equivalence and consequently their corresponding system matrix transformations of strict system equivalence and extended strict system equivalent do not preserve infinite zeros.

The remainder of this chaper will be devoted to a discussion of transformations which do preserve infinite zeros. Verghese (1978) studied the generalised state-space form of a system matrix in detail and in this chapter attempts will be made to quantify the transformation by which any system matrix can be reduced to the generalised state space form.

## Section (4.2) : Constant Transformations which Preserve Infinite Zeros

Gantmacher (1959), Rosenbrock (1974a) and Verghese (1978) investigated transformations on polynomial matrices which preserve both the finite and infinite zero structure and they all describe constant transformations which have this property. Gantmacher concentrates his work on singular pencils of matrices (as described in section (3.2)) whilst Rosenbrock and Verghese consider system matrices in the generalised state-space form.

The following definition is given by Gantmacher.

(2.1) :Definition: Let $A+sB$ and $A_1+sB_1$ be two pencils of rectangular matrices of dimensions mxn such that

$$P(A+sB)Q = A_1+sB_1 \tag{2.2}$$

where P and Q are constant non-singular matrices of dimensions mxm and nxn respectively. Then $A+sB$ and $A_1+sB_1$ are said to be STRICTLY EQUIVALENT (denoted s.e.).

Gantmacher shows that if two pencils are strictly equivalent then they have the same sets of finite and infinite elementary divisors (see definition (3.2.2.)) and that any matrix pencil can be transformed under strict equivalence to Kronecker form. In Kronecker form the finite and infinite elementary divisors and the minimal indices of the left and right null spaces of the pencil are displayed separately. As was shown in section (3.2) each kth order infinite elementary divisor corresponds to a (k-1) th order infinite zero of a pencil and consequently strict equivalence does also preserve the number of infinite zeros of the pencil.

Rosenbrock (1974a) applies strict equivalence to system matrices in the generalised state-space form as follows.

<u>(2.3) :Definition</u>: Two $(r+m)\times(r+\ell)$ system matrices

$$P(s) = \begin{bmatrix} sE-A & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix} \text{ and } P_1(s) = \begin{bmatrix} sE_1-A_1 & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \tag{2.4}$$

with

$$|sE-A| \not\equiv 0 \quad \text{and} \quad |sE_1-A_1| \not\equiv 0$$

are said to be related by RESTRICTED SYSTEM EQUIVALENCE (denoted r.s.e.) if

$$\begin{bmatrix} M & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_m \end{bmatrix} \begin{bmatrix} sE-A & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix} \begin{bmatrix} N & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_\ell \end{bmatrix} = \begin{bmatrix} sE_1-A_1 & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \tag{2.5}$$

where M, N are real non-singular matrices.

This transformation is a special case of strict system equivalence and therefore it clearly leaves invariant the transfer function matrix, the order n, the finite decoupling zeros and the finite system poles and system zeros.

Note that the matrix pencils $sE-A$ and $sE_1-A_1$ in (2.5) are s.e. Consequently, every system matrix in generalised state-space form can be reduced by r.s.e. to the form

$$P_2(s) = \left[\begin{array}{cc:c} sI_n-A_2 & 0 & B_2 \\ 0 & I_{r-n}+sJ & B_3 \\ \hdashline -C_2 & C_3 & D \end{array}\right] \tag{2.6}$$

where

$$\begin{bmatrix} sI_n-A_2 & 0 \\ 0 & I_{r-n}+sJ \end{bmatrix} \tag{2.7}$$

is the Kronecker form of the matrix $sE-A$ and $J$ is the Jordan normal form. The submatrix $(sI-A_2)$ in (2.7) relates to the finite zeros and the submatrix $(I_{r-n}+sJ)$ relates to the infinite elementary divisors of $sE-A$. Since $|sE-A| \not\equiv 0$, $sE-A$ has no left or right null spaces. $P_2(s)$ is said to be in standard form under r.s.e.

Rosenbrock gives a definition of infinite decoupling zeros which is different from definition (2.1.25) . i.e.

<u>(2.8) :Definition</u>: The Rosenbrock infinite i.d., o.d. and i.o.d. zeros of $P(s)$ are the i.d., o.d. and i.o.d. zeros respectively at $s=0$ of

$$\left[\begin{array}{c:c} E-sA & B \\ \hdashline -C & D \end{array}\right]. \tag{2.9}$$

Returning to the standard form (2.6), Rosenbrock shows that the finite decoupling zeros of $P(s)$ are the zeros of

$$\left[\begin{array}{c:c} sI_n-A_2 & B_2 \\ \hdashline -C_2 & D \end{array}\right] \tag{2.10}$$

and the Rosenbrock "infinite decoupling zeros" are the decoupling zeros at s=0 of

$$\begin{bmatrix} sI_{r-n}+J & \vdots & B_3 \\ \cdots & & \cdots \\ -C_3 & \vdots & D \end{bmatrix}. \tag{2.11}$$

The main problem with the definition (2.8) is that under it trivial expansion of the system matrix increases the number of Rosenbrock "decoupling zeros" as the next example shows. This is clearly an undesirable feature.

<u>(2.12) :Example:</u> Let

$$P(s) = \begin{bmatrix} s & \vdots & 1 \\ \cdots & & \cdots \\ 1 & \vdots & 0 \end{bmatrix} \quad \text{and } P_1(s) = \begin{bmatrix} s & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 0 \\ \cdots & & & \cdots \\ 1 & 0 & \vdots & 0 \end{bmatrix} \tag{2.13}$$

i.e. $P_1(s)$ is a trivial expansion of $P(s)$. Now

$$\begin{bmatrix} E-sA & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix} = \begin{bmatrix} 1 & \vdots & 1 \\ \cdots & & \cdots \\ 1 & \vdots & 0 \end{bmatrix} \tag{2.14}$$

and

$$\begin{bmatrix} E_1-sA & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & s & \vdots & 0 \\ \cdots & & & \cdots \\ 1 & 0 & \vdots & 0 \end{bmatrix} \tag{2.15}$$

Now (2.14) has no i.d. zeros at s=0 whereas (2.15) has one such i.d. zero. Therefore $P_1(s)$ has one infinite i.d. zero although $P(s)$ has none, even though $P_1(s)$ is just a trivial expansion of $P(s)$. Note that $P(s)$ and $P_1(s)$ are not r.s.e.

Clearly this definition of infinite decoupling zeros is unsatisfactory. Restricted system equivalence does preserve the infinite decoupling zeros as defined in

(2.6), but, more importantly, the infinite decoupling zeros as defined in (1.4) are also preserved. An obvious limitation of r.s.e. is that it does not permit trivial expansion of the system matrix.

Verghese (1978) has developed another definition of equivalence which he terms STRONG EQUIVALENCE. Under strong equivalence any operations of trivial expansion or contraction of the system matrix (which do not alter the transfer function matrix) are permitted in addition to the operations of r.s.e. For convenience Verghese works with the standard form under r.s.e. given by equation (2.6). When infinite frequency behaviour is being investigated only the submatrix

$$\left[\begin{array}{c:c} I_{r-n}-sJ & B_3 \\ \hdashline -C_3 & D \end{array}\right] \tag{2.16}$$

needs to be considered. Verghese points out that the 1x1 Jordan blocks have no significance for the free response of the system and thus he separates them out and writes

$$\left[\begin{array}{c:c} I_{r-n}-sJ & B_3 \\ \hdashline -C_3 & D \end{array}\right] = \left[\begin{array}{cc:c} I-s\hat{J} & 0 & \hat{B} \\ 0 & I & \check{B} \\ \hdashline -\hat{C} & -\check{C} & D \end{array}\right] \tag{2.17}$$

where $\hat{J}$ has no 1x1 Jordan blocks.

Under strong equivalence the system matrices

$$P_3(s) = \left[\begin{array}{cc:c} I_{r-n}-sJ & 0 & B_3 \\ 0 & I_k & B^* \\ \hdashline -C_3 & -C^* & D-C^*B^* \end{array}\right] \quad \text{for arbitrary } B^*, C^* \text{ and } k \tag{2.18}$$

and

$$P_4(s) \quad = \quad \begin{bmatrix} I-s\hat{J} & \vdots & \hat{B} \\ \cdots & & \cdots \\ -\hat{C} & \vdots & D+\check{C}\check{B} \end{bmatrix} \qquad (2.19)$$

are both equivalent to (2.17). The system matrix in (2.19) is important because it contains no non-dynamic internal variables. i.e. all 1x1 Jordan blocks have been removed. (2.19) is said to be in STANDARD FORM UNDER STRONG EQUIVALENCE.

The operations of strong equivalence are defined as follows.

(2.20) Definition: Two $(r+m)\times(r+\ell)$ generalised state-space system matrices $P(s)$ and $P_1(s)$ defined as in (2.4) are said to be related by OPERATIONS OF STRONG EQUIVALENCE if

$$\begin{bmatrix} L & \vdots & 0 \\ \cdots & & \cdots \\ M & \vdots & I_m \end{bmatrix}\begin{bmatrix} sE-A & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix}\begin{bmatrix} R & \vdots & N \\ \cdots & & \cdots \\ 0 & \vdots & I_\ell \end{bmatrix} = \begin{bmatrix} sE_1-A_1 & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \qquad (2.21)$$

where $L,M,N,R$ are constant matrices with $L$ and $R$ non-singular and

$$ME = EN = 0 \qquad (2.22)$$

Equation (2.21) is clearly a special case of (s.s.e.) and hence the transformation leaves invariant the transfer function matrix, the order $n$, the finite decoupling zeros and the finite system poles and zeros. By insisting that $L,M,N,R$ are constant the infinite frequency behaviour is also preserved. The extra requirement (2.22) ensures that the resulting matrix is in the generalised state-space form. Although it is not immediately obvious it can easily be shown that strong equivalence is a reflexive, symmetric and transitive relation and thus it is an equivalence relation. Of course, when $M=N=0$ (2.21) is simply restricted system equivalence. Clearly in (2.21) it is important that the system matrices $P(s)$ and $P_1(s)$ have the same dimensions. Strong equivalence is defined more generally as follows.

(2.23) :Definition: Two generalised state-space systems $S$ and $S_1$ with system matrices

$$P(s) = \begin{bmatrix} sE-A & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix} \text{ and } P_1(s) = \begin{bmatrix} sE_1-A_1 & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \qquad (2.24)$$

of dimensions $(r+m)\times(r+\ell)$ and $(r_1+m)\times(r_1+\ell)$ respectively are said to be STRONGLY EQUIVALENT (denoted str.eq.) if after some non-dynamic elimination or augmentation in either or both systems, the two are related by operations of strong equivalence.

Verghese suggests the following test to determine whether or not two generalised state-space systems are strongly equivalent.

(2.25) :Test: Two generalised state-space systems are strongly equivalent if and only if the respective standard forms for them, obtained by eliminating all non-dynamic variables, are related by operations of strong equivalence.

As required, matrices (2.17), (2.18) and (2.19) are all strongly equivalent to each other and thus strong equivalence is an improvement on restricted system equivalence. One disadvantage of strong equivalence is the phrase "after some non-dynamic elimination or augmentation in either or both systems" in definition (2.23). It would clearly be an improvement if such elimination or augmentation could be included as a matrix operation in the definition.

Returning to equation (2.21) and postmultiplying both sides by

$$\begin{bmatrix} R^{-1} & -R^{-1}N \\ 0 & I_\ell \end{bmatrix} \tag{2.26}$$

leads to

$$\begin{bmatrix} L & 0 \\ M & I_m \end{bmatrix}\begin{bmatrix} sE-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sE_1-A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix}\begin{bmatrix} R^{-1} & -R^{-1}N \\ 0 & I_\ell \end{bmatrix}. \tag{2.27}$$

The conditions (2.22) are no longer required since it is now implicit that

$$ME = E_1R^{-1}N = 0$$

holds.

In (2.27) the requirement that $P(s)$ and $P_1(s)$ have the same dimensions may also be relaxed. Of course the dimensions of the transforming matrices must be altered accordingly and certain other conditions come into play. The matrices L and "$R^{-1}$" are no longer square and thus equation (2.27) and the requirements that L and R be square and non-singular must be revised. In fact equation (2.27) is replaced by

$$\begin{bmatrix} L & 0 \\ M & I_m \end{bmatrix}\begin{bmatrix} sE-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sE_1-A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix}\begin{bmatrix} R' & N' \\ 0 & I_\ell \end{bmatrix} \tag{2.28}$$

with the matrices

$$(L \quad sE_1-A_1) \tag{2.29}$$

and

$$\begin{bmatrix} sE-A \\ R' \end{bmatrix} \tag{2.30}$$

having no finite or infinite zeros.

In (2.28) P(s) and $P_1(s)$ are e.s.s.e. (as in equation (1.3.15)) because of the requirements at finite s in (2.29) and (2.30) and hence the transformation preserves the finite frequency behaviour of the system. The requirement that (2.29) and (2.30) have no finite or infinite zeros ensures that $sE_1-A_1$ and sE-A also have the same infinite zeros as the next result shows.

<u>(2.31) Theorem</u>: Let P(s) and Q(s) with dimensions rxr and $r_1xr_1$ respectively be related by

$$L\,P(s) = Q(s)\,R. \tag{2.32}$$

Then P(s) and Q(s) have the same finite and infinite zeros if and only if the matrices

$$(L \quad Q(s)) \tag{2.33}$$

and

$$\begin{bmatrix} P(s) \\ R \end{bmatrix} \tag{2.34}$$

have no finite or infinite zeros.

<u>Proof</u>: Clearly P(s) and Q(s) are e.u.e. (and hence have the same finite zeros) if and only if (2.33) and (2.34) have no finite zeros.

Now P(s) and Q(s) have the same infinite zeros if and only if numerators of $P(\frac{1}{w})$ and $Q(\frac{1}{w})$ are also e.u.e. Let

$$P(\tfrac{1}{w}) = N_1(w)D_1^{-1}(w) \tag{2.35}$$

and

$$Q(\tfrac{1}{w}) = D_2^{-1}(w)N_2(w) \tag{2.36}$$

be relatively prime factorisations of $P(\frac{1}{w})$ and $Q(\frac{1}{w})$.

Thus

$$(L, Q(\tfrac{1}{w})) = D_2^{-1}(w)\ (D_2(w)L,\ N_2(w)) \tag{2.37}$$

and

$$\begin{bmatrix} P(\frac{1}{w}) \\ R \end{bmatrix} = \begin{bmatrix} N_1(w) \\ RD_1(w) \end{bmatrix} D_1^{-1}(w) \tag{2.38}$$

are also relatively prime factorisations.

Suppose first that (2.33) and (2.34) have no finite or infinite zeros and hence the numerators of (2.37) and (2.38) have full rank for all $w \in \mathbb{C}$.

Putting $s = \frac{1}{w}$ and substituting from (2.35) and (2.36) in (2.32) gives

$$LN_1(w)D_1^{-1}(w) = D_2^{-1}(w)N_2(w)R$$

i.e. $$D_2(w)\ L\ N_1(w) = N_2(w)\ R\ D_1(w). \tag{2.39}$$

Since the numerators of (2.37) and (2.38) have full rank the required relative primeness conditions are satisfied so that $N_1(w)$ and $N_2(w)$ are e.u.e. Hence they have the same zeros at w=0. Consequently P(s) and Q(s) have the same infinite zeros.

Conversely if P(s) and Q(s) have the same infinite zeros then their numerators are e.u.e. i.e. From equation (2.39) the matrices

$$(D_2(w)L,\ N_2(w))$$

and

$$\begin{bmatrix} N_1(w) \\ RD_1(w) \end{bmatrix}$$

have full rank at w=0. But these matrices are numerators of $(L, Q(\frac{1}{w}))$ and $\begin{bmatrix} P(\frac{1}{w}) \\ R \end{bmatrix}$ respectively. Hence (2.33) and (2.34) have no infinite zeros.

Thus sE-A and $sE_1-A_1$ in (2.28) have the same finite and infinite zeros and hence the transformation of equation (2.28) preserves both the finite and infinite frequency behaviour of the system. Pugh and Shelton (1979) show that e.s.s.e. includes trivial expansion in addition to the usual operations of s.s.e. In an analogous way the transformation of equation (2.28) is an extension of strong equivalence to include trivial expansion as a matrix operation in the definition. This transformation will be called EXTENDED STRONG EQUIVALENCE (denoted e.str.eq.) and defined as follows.

<u>(2.40) :Definition</u>: Two generalised state-space systems S and $S_1$ with system matrices P(s) and $P_1$(s) of dimensions (r+m) x(r+ℓ) and $(r_1+m) \times (r_1+\ell)$ respectively as in (2.24) are said to be extended strongly equivalent if and only if the systems matrices are related by

$$\begin{bmatrix} L & \vdots & 0 \\ \cdots & \cdots & \cdots \\ M & \vdots & I_m \end{bmatrix} \begin{bmatrix} sE-A & \vdots & B \\ \cdots & \cdots & \cdots \\ -C & \vdots & D \end{bmatrix} = \begin{bmatrix} sE_1-A_1 & \vdots & B_1 \\ \cdots & \cdots & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \begin{bmatrix} R & \vdots & N \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I_\ell \end{bmatrix} \tag{2.41}$$

where L,M,N' and R' are constant matrices with

$$(L,\ sE_1-A_1) \text{ and } \begin{bmatrix} sE-A \\ R' \end{bmatrix}$$

having no finite nor infinite zeros.

As would be expected the next theorem confirms that strong equivalence is a special case of extended strong equivalence.

<u>(2.42) :Theorem</u>: If two generalised state-space systems S and $S_1$ with $(r+m)\times(r+\ell)$ system matrices $P(s)$ and $P_1(s)$ are strongly equivalent then they are also extended strongly equivalent.

<u>Proof</u>: If $P(s)$ and $P_1(s)$ are related by strong equivalence then (2.21) holds with L and R constant and non-singular. Postmultiplying both sides of (2.21) by $\begin{bmatrix} R^{-1} & \vdots & -R^{-1}N \\ \cdots & \cdot & \cdots \\ 0 & \vdots & I_\ell \end{bmatrix}$

gives equation (2.27) which is clearly in the form of equation (2.41). Since L,R and consequently $R^{-1}$ are non-singular the matrices

$$\left[L,\ sE_1-A_1\right] \tag{2.43}$$

and

$$\begin{bmatrix} sE-A \\ R^{-1} \end{bmatrix} \tag{2.44}$$

have no finite zeros. Now $sE-A$ and $sE_1-A_1$ are strictly equivalent and they thus have the same infinite zeros. Thus, by theorem (2.31) the matrices (2.43) and (2.44) have no infinite zeros.

The matrices (2.17), (2.18) and (2.19) which Verghese showed to be strongly equivalent to each other are all related explicitly by extended strong equivalence. For example (2.17) and (2.19) are related by

$$\left[\begin{array}{cc:c} I & 0 & 0 \\ \hdashline 0 & \check{C} & I \end{array}\right]\left[\begin{array}{cc:c} I-s\hat{J} & 0 & \hat{B} \\ 0 & I & \check{C} \\ \hdashline -\hat{C} & -\check{C} & D \end{array}\right] = \left[\begin{array}{c:c} I-s\hat{J} & \hat{B} \\ \hdashline -\hat{C} & D+\check{C}\hat{B} \end{array}\right]\left[\begin{array}{cc:c} I & 0 & 0 \\ \hdashline 0 & 0 & I \end{array}\right]$$

which satisfies all the conditions of definition (2.40). Hence (2.17) and (2.19) are extended strongly equivalent. It can be shown similarly that (2.18) is extended strongly equivalent to both (2.17) and (2.19).

Thus the operations of trivial elimination and augmentation are directly included as matrix operations in definition (2.40) and hence this definition is preferable to definition (2.23). However, extended strong equivalence still has two important drawbacks. One is that it can only be applied to system matrices in generalised state-space form and not to polynomial system matrices of the form of (1.1). The second is that all the equivalence transformations developed in this section have involved only constant operations. The ultimate aim of this chapter is to develop a transformation by which any polynomial system matrix can be transformed into an equivalent system matrix in the generalised state-space form. It is obvious that no constant transformation will have this property and hence the transformations developed thus far will not be satisfactory. Clearly an 's' dependent transformation which preserves the structure of the system matrix at both finite and infinite frequencies is required and one such transformation will be described in the next section.

## Section (4.3) : Non-constant Transformations on Polynomial Matrices

In order to discuss s-dependent transformations on system matrices it is necessary to first consider non-constant transformations on polynomial matrices in general. In this section a transformation which preserves both the finite and infinite zero structure of a polynomial matrix will be described. It is known that if two polynomial matrices $P(s)$ and $P_1(s)$ are e.u.e. (definition (1.3.9)) then they have the same finite zero structure. In order to preserve the infinite zero structure the numerators of $P(\frac{1}{w})$ and $P_1(\frac{1}{w})$ must also be (e.u.e.). One relationship which satisfies these criteria is described in the following theorem.

(3.1) :Theorem: Let $P(s)$ and $P_1(s)$ be two polynomial matrices of dimensions $(r+m)\times(r+\ell)$ and $(r_1+m)\times(r_1+\ell)$ respectively. Then $P(s)$ and $P_1(s)$ have the same zeros, both finite and infinite, if

$$M(s)P(s) = P_1(s)N(s) \tag{3.2}$$

where $M(s)$ and $N(s)$ are polynomial matrices such that the matrices

$$(M(s),P_1(s)) \quad \text{and} \quad \begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}$$ possess no finite nor

infinite zeros and have McMillan degrees $\delta(P_1)$ and $\delta(P)$ respectively.

Proof: Since $(M(s), P_1(s))$ and $\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}$ have no finite zeros $P(s)$ and $P_1(s)$ are (e.u.e.) and hence they have the same finite zeros.

For the infinite zero structure consider the relatively prime factorisations

$$P(\tfrac{1}{w}) = \tilde{N}(P)\tilde{D}^{-1}(P) \tag{3.3}$$

and $$P_1(\tfrac{1}{w}) = \tilde{D}^{-1}(P_1)\tilde{N}(P_1). \tag{3.4}$$

Since $P_1(s)$ is polynomial it has no finite poles, merely infinite ones.

Thus $P_1(\frac{1}{w})$ has all its poles at $w=0$. Hence

$$\delta(P_1) = \nu(P_1(\tfrac{1}{w})). \tag{3.5}$$

Since (3.4) is a relatively left prime factorisation of the strictly proper matrix $P_1(\frac{1}{w})$

$$\delta(|\tilde{D}(P_1)|) = \nu(P_1(\tfrac{1}{w})). \tag{3.6}$$

Now suppose that

$$(M(\tfrac{1}{w}), P_1(\tfrac{1}{w})) = \tilde{D}^{-1}(M,P_1) \quad (\tilde{N}'(M), \tilde{N}'(P_1)) \tag{3.7}$$

and

$$\begin{bmatrix} P(\frac{1}{w}) \\ -N(\frac{1}{w}) \end{bmatrix} = \begin{bmatrix} \tilde{N}'(P) \\ \tilde{N}'(N) \end{bmatrix} \tilde{D}^{-1}\begin{bmatrix} P \\ N \end{bmatrix} \tag{3.8}$$

are relatively prime factorisations. Then

$$\begin{aligned} \delta(|\tilde{D}(M,P_1)|) &= \nu(M(\tfrac{1}{w}), P_1(\tfrac{1}{w})) \\ &= \delta(M(s), P_1(s)) \\ &= \delta(P_1) \end{aligned} \tag{3.9}$$

this last equality following from the hypothesis. Now from (3.5) and (3.9),

$$\delta(|\tilde{D}(M,P_1)|) = \nu(P_1(\tfrac{1}{w})) \tag{3.10}$$

and hence

$$\tilde{D}^{-1}(M,P_1)\tilde{N}'(P_1) = P_1(\tfrac{1}{w}) \tag{3.11}$$

is a relatively left prime factorisation of the strictly proper rational matrix $P_1(\frac{1}{w})$. i.e. $\tilde{N}'(P_1)$ is a numerator of $P_1(\frac{1}{w})$.

In an analogous manner $\tilde{N}'(P)$ from (3.8) is a numerator of $P(\frac{1}{w})$. Substituting (3.7) and (3.8) into (3.2) gives

$$(\tilde{N}'(M), \tilde{N}'(P_1)) \begin{bmatrix} \tilde{N}'(P) \\ \tilde{N}'(N) \end{bmatrix} = 0. \tag{3.12}$$

Now $(M(s), P_1(s))$ has no infinite zeros and therefore $(\tilde{N}'(M), \tilde{N}'(P_1))$ has full rank at w=0. In fact it has full rank for all w since $(M(s), P_1(s))$ has no finite nor infinite zeros from the hypothesis. Thus $\tilde{N}'(M)$ and $\tilde{N}'(P_1)$ are relatively left prime. In an analogous manner $\tilde{N}'(P)$ and $\tilde{N}'(N)$ are relatively right prime. Hence (3.12) is a relationship of (e.u.e.) between $\tilde{N}'(P_1)$ and $\tilde{N}'(P)$. Thus these matrices have the same finite zero structure. In particular $\tilde{N}'(P_1)$ and $\tilde{N}'(P)$ have the same zeros at w=0. i.e. $P_1(s)$ and $P(s)$ have the same infinite zeros.

In theorem (3.1) the requirements that $(M(s), P_1(s))$ and $\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix}$ have no finite zeros are both necessary and sufficient conditions for $P(s)$ and $P_1(s)$ to have the same finite zeros. However, the requirements that $(M(s), P_1(s))$ and $\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix}$ have no infinite zeros and the McMillan degree condition specified in the theorem are merely sufficient conditions to guarantee that the numerators of $P(\frac{1}{w})$ and $P_1(\frac{1}{w})$ are (e.u.e.) and hence that $P(s)$ and $P_1(s)$ have the same infinite zeros. It seems likely that these conditions could be relaxed somewhat in order to define only the necessary conditions for $P(s)$ and $P_1(s)$ to have the same infinite zeros but at present it is not clear how this might be done. The following example illustrates theorem (3.1) and in particular it demonstrates that merely requiring $(M(s), P_1(s))$ and $\begin{bmatrix} P_1(s) \\ N(s) \end{bmatrix}$ to have no finite or infinite zeros is not sufficient to ensure that $P(s)$ and $P_1(s)$ have the same infinite zeros.

(3.13):Example: Consider the matrices $M(s)$, $P(s)$, $P_1(s)$ and $N(s)$ such that

$$M(s)\ P(s) = P_1(s)\ N(s) \tag{3.14}$$

where

$$M(s) = \begin{bmatrix} s^3 & s^2 \\ 0 & 0 \end{bmatrix}, \qquad P(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_1(s) = \begin{bmatrix} s & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad N(s) = \begin{bmatrix} 0 & 0 \\ s^3 & s^2 \end{bmatrix}.$$

Now (3.14) is in the form of equation (3.2) and

$$(M(s), P_1(s)) = \begin{bmatrix} s^3 & s^2 & s & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.15}$$

has no finite or infinite zeros. Similarly

$$\begin{bmatrix} P(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ s^3 & s^2 \end{bmatrix} \tag{3.16}$$

also has no finite or infinite zeros. The McMillan degrees of these matrices are given by

$$\delta(M(s),P_1(s)) = 3 \quad ; \quad \delta(M(s)) = 3 \quad ; \quad \delta(P_1(s)) = 1$$

$$\delta\begin{bmatrix} P(s) \\ N(s) \end{bmatrix} = 3 \quad ; \quad \delta(P(s)) = 0 \quad ; \quad \delta(N(s)) = 3.$$

Thus the matrices M(s) and N(s) satisfy the McMillan degree conditions of theorem (3.1) and hence this theorem predicts that M(s) and N(s) have the same finite and infinite zeros. However, in the case of P(s) and $P_1(s)$ the McMillan degree conditions are not satisfied.

Relatively left prime factorisations of $M(\frac{1}{w})$, $P(\frac{1}{w})$, $P_1(\frac{1}{w})$ and $N(\frac{1}{w})$ are given by

$$M(\tfrac{1}{w}) = \begin{bmatrix} w^3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}$$

$$P(\tfrac{1}{w}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_1(\tfrac{1}{w}) = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w \\ 1 & 0 \end{bmatrix}$$

$$N(\tfrac{1}{w}) = \begin{bmatrix} 1 & 0 \\ 0 & w^3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & w \end{bmatrix}.$$

Thus M(s), N(s) and P(s) have no infinite zeros whereas $P_1(s)$ has one and hence, as predicted by theorem (3.1), M(s) and N(s) have the same infinite zeros. However, although neither (3.15) nor (3.16) has any finite or infinite zeros, P(s) and $P_1(s)$ do not have the same infinite zeros. Thus it is clear that either the McMillan degree condition of theorem (3.1) must be satisfied or some other requirement is necessary to ensure that $P(s)$ and $P_1(s)$ have the same infinite zeros.

The most important point concerning the transformation of theorem (3.1) is that it is s-dependent and thus a transformation of this type could possibly be used to linearise any polynomial matrix. However, it is not clear that this is an equivalence transformation. For example it is not known whether it is symmetric. i.e. if there exist matrices $M(s)$ and $N(s)$ such that $P(s)$ and $P_1(s)$ satisfy the requirements of theorem (3.1) with

$$M(s)P(s) = P_1(s)N(s)$$

it is not clear whether or not $M_1(s)$ and $N_1(s)$ with

$$M_1(s)P_1(s) = P(s)N_1(s)$$

also satisfy the requirements of the theorem.
Despite these disadvantages this transformation is important because it is s-dependent, unlike all the transformations which have been developed previously. One further attractive feature is that it includes the transformation which Gantmacher terms strict equivalence (definition (2.1)) as the following theorem establishes.

<u>(3.17):Theorem:</u> Let $P(s)$ and $P_1(s)$ be two mxn pencils of matrices. Then the transformation of (s.e.) namely

$$L\ P(s)\ R = P_1(s) \tag{3.18}$$

where L and R are non-singular constant matrices satisfies the hypothesis of theorem (3.1).

<u>Proof</u>: The transformation of (s.e.) of (3.14) may be written as

$$(L,\ P_1(s))\begin{bmatrix} P(s) \\ -R^{-1} \end{bmatrix} = 0. \qquad (3.19)$$

Consider the matrix

$$(L,\ P_1(s)). \qquad (3.20)$$

This clearly has full rank for all finite s since L is non-singular and constant. i.e. (3.16) has no finite zeros. To see that (3.20) has no infinite zeros let

$$P_1(\tfrac{1}{w}) = \tilde{D}_1^{-1}(w)\tilde{N}_1(w) \qquad (3.21)$$

be a relatively left prime factorisation of $P_1(\frac{1}{w})$. Then

$$\tilde{D}_1^{-1}(w)\,(\tilde{D}_1(w)L,\ \tilde{N}_1(w)) \qquad (3.22)$$

is a relatively left prime factorisation of

$$(L,\ P_1(\tfrac{1}{w})). \qquad (3.23)$$

Thus $(\tilde{D}_1(w)L,\ \tilde{N}_1(w))$ is a numerator of (3.23) which has the same zeros for finite w as

$$(\tilde{D}_1(w),\ \tilde{N}_1(w)). \qquad (3.24)$$

In particular (3.24) has no zeros at $w = 0$, due to the relative primeness of the factorisation (3.21). Thus (3.20) has no infinite zeros.

Finally, the McMillan degree of $(L,\ P_1(s))$ is clearly identical to that of $P_1(s)$, since the minors of (3.20) are either exactly minors of $P_1(s)$ or else constant linear combinations of these.

In a similar way the matrix

$$\begin{bmatrix} P(s) \\ -R^{-1} \end{bmatrix} \qquad (3.25)$$

can be shown to have no finite or infinite zeros and to have McMillan degree identical to that of P(s). The theorem thus follows.

(3.26):Corollary: The transformation of (s.e.) preserves the finite and infinite zero structure of a polynomial matrix.

Proof: This follows immediately from theorems (3.1) and (3.17).

In this section an s-dependent transformation which preserves both the finite and infinite zeros of a polynomial matrix has been presented. Unfortunately the transformation describes only sufficient conditions for two polynomial matrices to have the same finite and infinite zeros and of course it would be a considerable improvement if the precise necessary conditions could be specified but this problem remains unsolved at present. Another disadvantage of this transformation is that it is probably not an equivalence transformation. However, despite these disadvantages, the transformation of theorem (3.1) represents an important step forward since it is s-dependent. All the infinite zero preserving transformations which have been described previously have involved only constant operations and thus it has obviously not been possible to use such transformations to linearise a polynomial matrix. The final example in this section illustrates the use of theorem (3.1) in the linearisation of a polynomial matrix.

(3.27): Example: Let

$$P(s) = \begin{bmatrix} s^2 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$P_1(s) = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $P(s)$ and $P_1(s)$ can be related by the equation

$$M(s)\ P(s) = P_1(s)N(s)$$

where $M(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix}$ and $N(s) = \begin{bmatrix} 0 & 1 \\ s & 0 \\ 0 & 0 \end{bmatrix}$.

The matrices $(M(s), P_1(s))$ and $\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}$ have no finite or infinite zeros and

$$\delta(M(s), P_1(s)) = \delta(P_1(s)) = 2$$

and $$\delta\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix} = \delta(P(s)) = 2.$$

Thus the conditions of theorem (3.1) are satisfied and hence, as can easily be seen by direct examination, $P(s)$ and $P_1(s)$ have the same finite and infinite zeros. It is clear therefore that $P_1(s)$ is a linearisation of $P(s)$ and that $P(s)$ is related to its linearised form by the transformation of theorem (3.1).

The question of linearisation will be considered further in section 5 but first, in the next section, the transformation on polynomial matrices described in this section will be applied to system matrices.

Section (4.4): Equivalence of System Matrices

In this section the transformation of theorem (3.1) will be applied to system matrices in the normalised form. Of course the generalised state-space form is a special case of the normalised form. At present it is not clear how to apply this transformation to polynomial system matrices in general without first expanding them to the normalised form.

Accordingly consider two systems described by normalised system matrices

$$P(s) = \begin{bmatrix} T(s) & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix} \quad \text{and } P_1(s) = \begin{bmatrix} T_1(s) & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \tag{4.1}$$

where $B, C, D, B_1, C_1$ and $D_1$ are constant matrices. Now let

$$M(s)T(s) = T_1(s)N(s) \tag{4.2}$$

for some polynomial matrices $M(s)$ and $N(s)$ such that

$(M(s),\ T_1(s))$ and $\begin{bmatrix} T(s) \\ -N(s) \end{bmatrix}$ have no finite or infinite

zeros,

$$\delta(M(s),\ T_1(s)) = \delta(T_1(s)) \tag{4.3}$$

and

$$\delta\begin{bmatrix} T(s) \\ -N(s) \end{bmatrix} = \delta(T(s)). \tag{4.4}$$

i.e. $T(s)$ and $T_1(s)$ are related by the transformation of theorem (3.1) and hence they have the same set of finite and infinite zeros. Now let

$$\begin{bmatrix} M(s) & \vdots & O \\ \cdots & & \cdots \\ X & \vdots & I_m \end{bmatrix} \begin{bmatrix} T(s) & \vdots & B \\ \cdots & & \cdots \\ -C & \vdots & D \end{bmatrix} = \begin{bmatrix} T_1(s) & \vdots & B_1 \\ \cdots & & \cdots \\ -C_1 & \vdots & D_1 \end{bmatrix} \begin{bmatrix} N(s) & \vdots & Y \\ \cdots & & \cdots \\ O & \vdots & I_\ell \end{bmatrix} \tag{4.5}$$

where X and Y are constant. Then

<u>(4.6):Theorem</u>: $P(s)$ and $P_1(s)$ are related by the transformation of theorem (3.1).

<u>Proof</u>: Let

$$Q_1(s) = \begin{bmatrix} M(s) & O & T_1(s) & B_1 \\ X & I_m & -C_1 & D_1 \end{bmatrix} . \tag{4.7}$$

Clearly $Q_1(s)$ has no finite zeros since $(M(s),\ T_1(s))$ has no finite zeros.

In order to consider the infinite zeros of $Q_1(s)$ let

$$(M(\tfrac{1}{w}),\ T_1(\tfrac{1}{w})) = L^{-1}(w)\ \ (L(w)M(\tfrac{1}{w}),\ L(w)T_1(\tfrac{1}{w})) \tag{4.8}$$

be a relatively left prime factorisation. Since $(M(s), T_1(s))$ has no infinite zeros the num erator of (4.8) has full rank at $w = O$. Now

$$Q_1(\tfrac{1}{w}) = \begin{bmatrix} M(\tfrac{1}{w}) & O & T_1(\tfrac{1}{w}) & B_1 \\ X & I_m & -C_1 & D_1 \end{bmatrix}$$

$$= \begin{bmatrix} L(w) & O \\ O & I_m \end{bmatrix}^{-1} \begin{bmatrix} L(w)M(\tfrac{1}{w}) & O & L(w)T_1(\tfrac{1}{w}) & L(w)B_1 \\ X & I_m & -C_1 & D_1 \end{bmatrix} \tag{4.9}$$

from (4.8). Since (4.8) is a relatively left prime factorisation (4.9) is also a relatively left prime factorisation and the numerator has full rank at $w = O$. Consequently $Q_1(s)$ has no infinite zeros.

The second block row of $Q_1(s)$ is constant and B is constant. Therefore, the minors of $Q_1(s)$ are either constant or, if they are s-dependent, they will be identical to or equal to constant linear combinations of the minors of $(M(s)\ T_1(s))$. Hence

$$\begin{aligned} \delta(Q_1(s)) &= \delta(M(s),\ T_1(s)) \\ &= \delta(T_1(s)) \end{aligned}$$

from (4.3). By a similar argument

$$\delta\begin{bmatrix} T_1(s) & B_1 \\ -C_1 & D_1 \end{bmatrix} = \delta(T_1(s))$$

and hence

$$\delta(P_1(s)) = \delta(Q_1(s)).$$

Similarly it can be shown that the matrix

$$Q(s) = \begin{bmatrix} T(s) & B \\ -C & D \\ N(s) & Y \\ 0 & I_\ell \end{bmatrix}$$

has no finite or infinite zeros and that

$$\delta(Q(s)) = \delta(P(s)).$$

Thus the theorem follows.

Two systems whose system matrices satisfy equations (4.1) to (4.5) inclusive will be said to be related by the transformation of NORMALISED SYSTEM EQUIVALENCE (written (n.s.e.)). The next theorem shows the importance of this transformation.

<u>(4.10):Theorem</u>: The transformation of (n.s.e.) preserves

(i) the transfer function matrix and hence its least order $\nu(G)$.

(ii) the sets of finite and infinite system poles.

(iii) all sets of finite and infinite decoupling zeros.

(iv) the sets of finite and infinite system zeros.

Proof: The transformation of (n.s.e.) is a special case of (e.s.s.e.) and hence, by theorem (1.3.19), it preserves the transfer function matrix, the set of finite system poles, all sets of finite decoupling zeros and the set of finite system zeros.

(ii) Recall equation (1.15) in which the system poles were defined as the zeros of $T(s)$. From equations (4.1) - (4.4) $T(s)$ and $T_1(s)$ have the same zeros, both finite and infinite and hence $P(s)$ and $P_1(s)$ have the same set of finite and infinite system poles.

(iii) Under (n.s.e.) the input-decoupling zeros are transformed according to

$$M(s)\ (T(s)\ ,\ B) = (T_1(s),\ B_1)\begin{bmatrix} N(s) & Y \\ O & I_\ell \end{bmatrix}. \tag{4.11}$$

From (1.3.19) the finite i.d. zeros are preserved and hence

$$(M(s),\ T_1(s),\ B_1) \tag{4.12}$$

has full rank for all finite s. Now let

$$(M(\tfrac{1}{w}),\ T_1(\tfrac{1}{w}),\ B_1) = L^{-1}(w)\,(L(w)M(\tfrac{1}{w}),\ L(w)T_1(\tfrac{1}{w}), L(w)B_1) \tag{4.13}$$

where $L(w)$ is a denominator of $(M(\frac{1}{w}),\ T_1(\frac{1}{w}))$ as in equation (4.8).

Clearly, from (4.8), (4.13) is a relatively left prime factorisation and the numerator has full rank at $w = 0$. Hence (4.12) has no infinite zeros. Since $B_1$ is constant the minors of (4.12) are either constant or, if they are s-dependent they are identical to or equal to constant linear combinations of the minors of $(M(s),\ T_1(s))$. Consequently

$$\delta(M(s),\ T_1(s),\ B) = \delta(M(s),\ T_1(s))$$

$$= \delta(T_1(s)) \quad \text{from (4.3)}$$

$$= \delta(T_1(s),\ B)$$

by a similar argument.

Similarly it can be shown that the matrix

$$\begin{bmatrix} N(s) & Y \\ O & I_\ell \\ T(s) & B \end{bmatrix}$$

has no finite or infinite zeros and has McMillan degree equal to that of $(T(s), B)$.

Thus the matrices in (4.11) satisfy the requirements of theorem (3.1) and hence $(T(s), B)$ and $(T_1(s), B_1)$ have the same finite and infinite zeros. i.e. $P(s)$ and $P_1(s)$ have the same finite and infinite i.d. zeros.

The invariance of the other sets of decoupling zeros under this transformation can be proved in a similar manner.

Considering the point $s_o = \infty$ in equation (1.23(b)) gives

$$\left\{\begin{matrix}\text{Infinite}\\ \text{system}\\ \text{zeros}\end{matrix}\right\} = \left\{\begin{matrix}\text{Infinite}\\ \text{transfer}\\ \text{function}\\ \text{zeros}\end{matrix}\right\} + \left\{\begin{matrix}\text{Infinite}\\ \text{i.d.}\\ \text{zeros}\end{matrix}\right\} + \left\{\begin{matrix}\text{Infinite}\\ \text{o.d.}\\ \text{zeros}\end{matrix}\right\} - \left\{\begin{matrix}\text{Infinite}\\ \text{i.o.d.}\\ \text{zeros}\end{matrix}\right\} \tag{4.13}$$

Since the transfer function matrix is not altered by this transformation the infinite transfer function zeros are invariant. Also this transformation preserves all sets of infinite decoupling zeros and hence the righthand side of equation (4.13) is not altered by (n.s.e.) Hence the infinite system zeros are invariant under the transformation.

The importance of this transformation is that it is s-dependent and further that it includes extended strong equivalence as the next theorem shows.

(4.14):Theorem: If two generalised state-space system matrices $P(s)$ and $P_1(s)$ are extended strongly equivalent then they are also (n.s.e.).

Proof: From definition (2.40) $P(s)$ and $P_1(s)$ are said to be extended strongly equivalent if and only if

$$\begin{bmatrix} L & O \\ M & I_m \end{bmatrix} \begin{bmatrix} sE-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sE_1-A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} \begin{bmatrix} R' & N' \\ O & I_\ell \end{bmatrix} \qquad (4.15)$$

where $L, M, N'$ and $R'$ are constant matrices with

$$(L,\ sE_1-A_1)$$

and

$$\begin{bmatrix} sE-A \\ R' \end{bmatrix}$$

having no finite or infinite zeros. Clearly the only additional requirement for $sE-A$ and $sE_1-A_1$ to be related by the transformation of theorem (3.1) is that the McMillan degree condition must be satisfied.

Now all the minors of $(L,\ sE_1-A)$ are either equal to or linear combinations of the minors of $sE_1-A_1$. Thus

$$\delta(L, sE_1-A_1) = \delta(sE_1-A_1).$$

By a similar argument

$$\delta \begin{bmatrix} sE-A \\ R' \end{bmatrix} = \delta(sE-A)$$

and thus $sE-A$ and $sE_1-A_1$ are related by the transformation of theorem (3.1). Hence, by theorem (4.6), $P(s)$ and $P_1(s)$ are (n.s.e.).

Thus the transformation of (n.s.e.) is seen to include all the constant transformations on system matrices that were described in section (4.2).

## Section (4.5): Transformation of a Polynomial System Matrix to the Generalised State-Space Form.

As was stated earlier in this chapter many authors (e.g. Wolovich (1974), Rosenbrock (1970)) have described algorithms by which any polynomial system matrix

$$P_1(s) = \begin{bmatrix} T_1(s) & U(s) \\ -V_1(s) & W(s) \end{bmatrix} \qquad (5.1)$$

can be realised in the state-space form

$$P_2(s) = \begin{bmatrix} sI-A & B \\ -C & D(s) \end{bmatrix} \qquad (5.2)$$

where $D(s)$ may be constant or even zero. In corollary (4.1.29) it was shown that $P_2(s)$ has no infinite decoupling zeros and hence when infinite frequency behaviour is being considered $P_2(s)$ must be replaced by the system matrix in the generalised state space form

$$P_3(s) = \begin{bmatrix} sE-A & B \\ -C & D(s) \end{bmatrix} \qquad (5.3)$$

where $E$ may be singular. In fact, by theorem (4.1.28), $E$ must be singular if $P_3(s)$ has any infinite decoupling zeros. Thus it is desirable to find a method by which any polynomial system matrix $P_1(s)$ may be transformed to the generalised state-space form P3(s) without altering the transfer function matrix and the finite and infinite decoupling zeros.

Some attempts that have been made to solve this problem will be discussed in this section. Godbout and Jordan (1975) and Gohberg, Lancaster and Rodman (1976) describe transformations by which a polynomial matrix L(s) may be linearised in the form sE-A. Godbout and Jordan then go on to describe further steps by which the generalised state-space form for the entire system matrix may be achieved. However, neither paper considers the infinite frequency behaviour of the system. It was hoped that these transformations could be adapted so that the infinite frequency characteristics of the system could also be preserved although, as will be seen, this has not been possible to achieve.

Gohberg et al (ibid) describe the linearisation of a matrix L(s) of degree $\ell$ (where in this sense the "degree of L(s)" refers to the highest degree of any element occurring in L(s)) according to an equation of the form

$$L(s) \oplus I_{\ell-1} = E(s)(sI-T)F(s) \qquad (5.4)$$

where $\oplus$ denotes the direct sum, sI-T is the linearised form of L(s) and E(s) and F(s) are polynomial matrices. This method is only valid for square monic L(s). In order to remove the restriction that L(s) must be square both sides of (5.4) may be post multiplied by $F^{-1}(s)$ and only the first block row of the resulting equation considered. This leads to an equation of the form

$$L(s)F_1(s) = E_1(s)(sE^*-A^*). \tag{5.5}$$

The exact form of the matrices in equation (5.5) is described in Appendix 1. $L(s)$ and $(sE^*-A^*)$ are extended unimodular equivalent and hence they have the same finite zeros. It is not at once clear whether or not they also have the same set of infinite zeros but the next example shows that this is not the case.

<u>(5.6):Example</u>: Let

$$L(s) = \begin{bmatrix} 1+s^2 & s+s^3 \\ 1+s^2 & 1+s^3 \end{bmatrix} \tag{5.7}$$

so that a relatively right prime factorisation of $L(\frac{1}{w})$ is given by

$$L(\tfrac{1}{w}) = \begin{bmatrix} 0 & 1+w^2 \\ 1-w & 1+w^3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -w & w^3 \end{bmatrix}^{-1}. \tag{5.8}$$

The Smith form of the numerator of (5.8) is

$$\begin{bmatrix} 1 & 0 \\ 0 & (1-w)(1+w^2) \end{bmatrix}$$

and thus $L(s)$ has no infinite zeros.

Following the method outlined in appendix 1 $L(s)$ may be realised in the form

$$sE^*-A^* = \begin{bmatrix} s & 0 & -1 & 0 & 0 & 0 \\ 0 & s & 0 & -1 & 0 & 0 \\ 0 & 0 & s & 0 & -1 & -1 \\ 0 & 0 & 0 & s & -1 & -1 \\ 1 & 0 & 0 & 1 & 1 & s \\ 1 & 1 & 0 & 0 & 1 & s \end{bmatrix}$$

Now a relatively right prime factorisation of $\frac{1}{w}E^*-A^*$ is given by

$$\frac{1}{w}E^*-A^* = \begin{bmatrix} 1 & 0 & w & 0 & 0 & 0 \\ 0 & 1 & 0 & -w & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -w \\ w & 0 & 0 & w & 1 & 1 \\ w & w & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & w \end{bmatrix}^{-1}$$

and the numerator has Smith form

$$\begin{bmatrix} I_5 & 0 \\ 0 & w^2(1+w^2)(w-1) \end{bmatrix} .$$

Hence $sE^*-A^*$ has two infinite zeros whereas $L(s)$ had none and thus the transformation has introduced 2 additional infinite zeros.

It is clear that the transformation of equation (5.5) does not preserve the infinite zero structure of $L(s)$. This transformation may be modified by considering each of the columns of $L(s)$ separately. The resulting transformation, which is described in detail in appendix 2, amounts to the standard method by which second and higher order ordinary differential equations are expanded to form first order state equations. This is also part of the transformation described by Godbout and Jordan (1975).

However, although this transformation introduces fewer additional infinite zeros than the transformation of equation (5.5) it still does not preserve the infinite zero structure of L(s) in all cases as the next example, which is a continuation of example (5.6) illustrates.

<u>(5.9):Example</u>: Consider again

$$L(s) = \begin{bmatrix} 1+s^2 & s+s^3 \\ 1+s^2 & 1+s^3 \end{bmatrix}$$

which has no infinite zeros. Treating the columns of L(s) separately L(s) may be realised as

$$sE'-A' = \begin{bmatrix} s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & s & -1 \\ 1 & s & 0 & 1 & s \\ 1 & s & 0 & 1 & s \end{bmatrix}$$

and a relatively right prime factorisation of $\frac{1}{w}E'-A'$ is given by

$$\frac{1}{w}E'-A' = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -w & 0 \\ 0 & 1 & 0 & 1 & -w \\ w & 0 & 0 & w & 1 \\ w & 0 & w & 0 & 1 \end{bmatrix} \begin{bmatrix} w & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 0 \\ 0 & -1 & 0 & 0 & w \end{bmatrix}^{-1}$$

where the numerator has Smith form

$$\begin{bmatrix} I_4 & 0 \\ 0 & w(1-w)(1+w^2) \end{bmatrix} .$$

Thus $sE'-A'$ has one infinite zero whereas L(s) had none and hence the transformation has not preserved the infinite zero structure of L(s).

Thus, although using the transformation in which the columns of $L(s)$ are considered separately introduces less additional infinite zeros than the transformation of equation (5.5) it is obviously not satisfactory. Hence it appears that no modification of the transformations described by Godbout and Jordan (1975) and Gohberg et al (1976) is going to lead to a suitable transformation for linearising $L(s)$ and preserving its infinite zero structure and consequently it is clear that a completely different approach is required.

This problem has recently been solved by Bosgra and Van der Weiden (1981) who describe an algorithm that brings any polynomial system matrix to the generalised state-space form while preserving the transfer function matrix and the finite and infinite decoupling zeros.

## Chapter 5. Feedback Considerations

### Section (5.1) : Constant Output Feedback and System Decoupling Zeros

The effect of constant output feedback on the finite poles and zeros of the transfer function matrix and on the finite decoupling zeros of the system has been discussed by various authors and the results are well-documented. In particular, it is known (Rosenbrock 1970) that the finite zero structure of a given rational transfer function matrix is completely unchanged by such action, although the finite pole structure possesses no such invariant property. Rosenbrock also shows that the finite decoupling zeros of the system are not altered by the application of constant output feedback. This chapter will be devoted to determining whether or not this state of affairs persists when the infinite poles and zeros are considered. In the present section the infinite decoupling zeros will be discussed, whilst the infinite poles and zeros of the transfer function matrix are examined in the next section.

Consider a system described, after Laplace transformation with zero initial conditions by the matrix equation

$$\begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ -\bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y} \end{bmatrix} \qquad (1.1)$$

where $T(s)$, $U(s)$, $V(s)$ and $W(s)$ are polynomial matrices of dimensions $r \times r$, $r \times \ell$, $m \times r$ and $m \times \ell$ respectively and the system matrix is given by

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} . \qquad (1.2)$$

The $m \times \ell$ transfer function matrix G(s) is, of course, given by

$$G(s) = V^{-1}(s)T(s)U(s)+W(s).$$

Now let constant output feedback, summarised by the $\ell \times m$ matrix F, be applied according to the equation

$$\bar{u} = \bar{v} - F\bar{y} \qquad (1.3)$$

so that, combining (1.1) and (1.3) gives the matrix equation for the system with constant output feedback

$$\left[\begin{array}{ccc:c} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & -I_m & 0 \\ 0 & I_\ell & F & -I_\ell \\ \hdashline 0 & 0 & I_m & 0 \end{array}\right] \left[\begin{array}{c} \xi \\ -\bar{u} \\ -\bar{y} \\ \hdashline -\bar{v} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline \bar{y} \end{array}\right]$$

where the feedback system matrix is given by

$$P_F(s) = \left[\begin{array}{ccc:c} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & -I_m & 0 \\ 0 & I_\ell & F & -I_\ell \\ \hdashline 0 & 0 & I_m & 0 \end{array}\right] \qquad (1.4)$$

The next result shows that P(s) and $P_F(s)$ have the same sets of finite and infinite decoupling zeros.

<u>(1.5):Theorem</u>: The finite and infinite input-decoupling, output-decoupling and input-output-decoupling zeros of a system are invariant under constant output feedback.

<u>Proof</u>: The finite cases were proved by Rosenbrock (1970 p. 157). Following theorem (4.1.4) the infinite input-decoupling zeros of P(s) are the zeros at w=0 of

$$Z(\tfrac{1}{w}) = \begin{bmatrix} T(\frac{1}{w}) & U(\frac{1}{w}) & 0 \\ -V(\frac{1}{w}) & W(\frac{1}{w}) & -I_m \end{bmatrix}.$$

Let a relatively left prime factorisation of $Z(\frac{1}{w})$ be given by $Z(\frac{1}{w}) = D_1^{-1}(w)N_1(w)$

so that the infinite input-decoupling zeros of P(s) are the zeros at w=0 of $N_1(w)$.

Similarly the infinite input-decoupling zeros of $P_F(s)$ are the zeros at w=0 of

$$Z_F(\tfrac{1}{w}) = \left[\begin{array}{ccc:c:c} T(\frac{1}{w}) & U(\frac{1}{w}) & 0 & 0 & 0 \\ -V(\frac{1}{w}) & W(\frac{1}{w}) & -I_m & 0 & 0 \\ 0 & I_\ell & F & -I_\ell & 0 \\ \hdashline 0 & 0 & I_m & 0 & -I_m \end{array}\right]$$

$$= \begin{bmatrix} D_1(w) & 0 & 0 \\ 0 & I_\ell & 0 \\ 0 & 0 & I_m \end{bmatrix}^{-1} \left[\begin{array}{c:c:c} N_1(w) & 0 & 0 \\ (0, I_\ell, F) & -I_\ell & 0 \\ (0,0,I_m) & 0 & -I_m \end{array}\right] \quad (1.6)$$

$$= D_2^{-1}(w)N_2(w).$$

Now, since $D_1(w)$ and $N_1(w)$ are relatively left prime (1.6) is a relatively left prime factorisation of $Z_F(\frac{1}{w})$, i.e. the infinite input-decoupling zeros of $P_F(s)$ are the zeros at w=0 of $N_2(w)$.

However, $N_1(w)$ and $N_2(w)$ are related by the equation

$$N_1(w)\,(I_{m+\ell} \,\vdots\, 0, \,\vdots\, 0) = (I_{m+\ell}, 0, 0) \left[\begin{array}{c:c:c} N_1(w) & 0 & 0 \\ (0, I_\ell, F) & -I_\ell & 0 \\ (0,0,I_m) & 0 & -I_m \end{array}\right]$$

and the matrices

$(N_1(w), I_{m+\ell}, 0, 0)$

and

$$\left[\begin{array}{c:c:c} I_{m+\ell} & 0 & 0 \\ N_1(w) & 0 & 0 \\ (0,I_\ell,F) & -I_\ell & 0 \\ (0,0,I_m) & 0 & -I_m \end{array}\right]$$

have full row and column rank respectively. Hence $N_1(w)$ and $N_2(w)$ are extended unimodular equivalent (see section (1.4)) and thus they have the same zeros at w=0, i.e. P(s) and $P_F(s)$ have the same set of infinite input-decoupling zeros.

The results for the infinite output-decoupling zeros and consequently the infinite input-output-decoupling zeros may be proved similarly.

## Section (5.2): The Effect of Constant Output Feedback on the Poles and Zeros of the Transfer Function Matrix

In the last section constant output feedback was discussed in terms of its effect on the system matrix and on the decoupling zeros. In this section its effect on the poles and zeros of the transfer function matrix will be considered. Many of the results presented here have been published in Pugh and Ratcliffe (1979b) and (1980). The application of constant output feedback as outlined in the last section is described in terms of the transfer function matrix by figure (2.1).

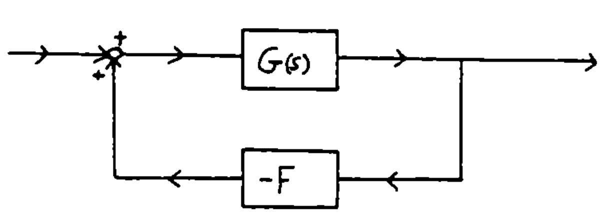


Figure (2.1).

If $G_F(s)$ denotes the transfer function matrix of the feedback system then

$$G_F(s) = G(s)(I_\ell+FG(s))^{-1}$$
$$= (I_m+G(s)F)^{-1}G(s) \qquad (2.2)$$

provided that, as will be assumed in the remainder of this section,

$$|I_\ell+FG(s)| \equiv |I_m+G(s)F|$$
$$\neq 0. \qquad (2.3)$$

In section (3.4) a special type of relatively prime factorisation of any rational matrix G(s) which was termed a minimal factorisation was described. Minimal factorisations were shown to be important in that besides displaying the finite poles and zeros of G(s) they also display the infinite poles and zeros in a particularly simple way. It is this property that forms an important part of the proof of the theorem concerning the effect of output feedback on the poles and zeros of the transfer function matrix that will be given here. As a first result in this direction the following theorem indicates that minimal factorisations of the open loop transfer function matrix G(s) are closely related to minimal factorisations of the feedback transfer function matrix $G_F(s)$.

<u>(2.4):Theorem</u>: If

$$G(s) = D(s)^{-1}N(s) \qquad (2.5)$$

is a minimal factorisation of G(s) then

$$G_F(s) = (D(s)+N(s)F)^{-1}N(s) \qquad (2.6)$$

is a minimal factorisation of $G_F(s)$.

Proof: Substituting for G(s) from (2.5) in (2.2) gives

$$G_F(s) = (I_m+D^{-1}(s)N(s)F)^{-1}D^{-1}(s)N(s)$$
$$= (D(s)+N(s)F)^{-1}N(s) \qquad (2.7)$$

Hence (2.6) is clearly a polynomial factorisation of $G_F(s)$ but as yet it is not clear whether or not it is also a minimal factorisation.

Now

$$(D(s)+N(s)F,\ N(s)) = (D(s),N(s)) \begin{bmatrix} I_m & O \\ F & I_\ell \end{bmatrix}. \qquad (2.8)$$

Since (D(s), N(s)) forms a minimal basis it has in particular full row rank for all finite $s\in\mathbb{C}$. Consequently, from (2.8) it follows that (D(s)+N(s)F, N(s)) also has full row rank for all finite $s\in\mathbb{C}$, which of course implies that the factorisation (2.7) is relatively left prime.

Let

$$(d_i(s),n_i(s)) \qquad (2.9)$$

denote the ith row of (D(s),N(s)) so that the ith row of (D(s)+N(s)F,N(s)) is

$$(d_i(s)+n_i(s)F,\ n_i(s)). \qquad (2.10)$$

If $\delta(n_i) < \delta(d_i)$ then the degree of (2.10) is clearly $\delta(d_i)$ since F is a constant matrix. But in this case $\delta(d_i)$ is the degree of (2.9). Hence (2.9) and (2.10) have the same degree.

If on the other hand $\delta(n_i) \geqslant \delta(d_i)$ then, since F is constant,

$$\delta(d_i+n_iF) \leqslant \delta(n_i). \qquad (2.11)$$

However, the last $\ell$ columns of (2.10) are still $n_i(s)$ so the degree of (2.10) is precisely $\delta(n_i)$. But $\delta(n_i)$ in this case is the degree of (2.9) and so again (2.9) and (2.10) have the same degree.

It thus follows that $(D(s)+N(s)F,\ N(s))$ and $(D(s),N(s))$ have the same row degrees. Consequently from (2.8),

$$[D+NF,\ N]_h = [D,N]_h \begin{bmatrix} I_m & 0 \\ F & I_\ell \end{bmatrix} \tag{2.12}$$

where $[\ \ ]_h$ denotes the high order coefficient matrix of the indicated matrix. Now $(D(s),\ N(s))$ is a minimal basis and so from definition (3.3.1(ii))its high order coefficient matrix $[D,N]_h$ has full row rank. Hence, from (2.12), the high order coefficient matrix of $(D(s)+N(s)F,\ N(s))$ also has full row rank and so $(D(s)+N(s)F,\ N(s))$ forms a minimal basis as required.

This result, in view of theorem (3.4.1), gives immediately,

<u>(2.13):Theorem</u>: Let $\delta_i$ $(i = 1,2,\ \ldots.,\ m)$ denote the row degrees of

$$(D(s),\ N(s))$$

where

$$G(s) = D^{-1}(s)N(s)$$

is a minimal factorisation, and let

$$\Lambda(w) = \mathrm{diag}(w^{\delta_1}, w^{\delta_2},\ \ldots.,\ w^{\delta_m}).$$

Then,

(i) the finite poles of $G_F(s)$ are the finite zeros of $(D(s)+N(s)F)$ and the infinite poles of $G_F(s)$ are the zeros at $w=0$ of $\Lambda(w)(D(\frac{1}{w})+N(\frac{1}{w})F)$.

(ii) the finite zeros of $G_F(s)$ are the finite zeros of $N(s)$ and the infinite zeros of $G_F(s)$ are the zeros at $w=0$ of $\Lambda(w)N(\frac{1}{w})$.

It is clear from theorems (2.4) and (2.13) that $N(s)$ is a numerator from minimal factorisations of both $G(s)$ and $G_F(s)$. Hence

(2.14):Theorem: The finite and infinite zeros of the transfer function matrix are invariant under constant output feedback.

This theorem confirms Rosenbrock's (1970) result for the finite zeros whilst the generalisation of the result to include the infinite zeros is new.

From theorems (2.4) and (2.13) it can be seen that whilst $D(s)$ is a denominator from a minimal factorisation of $G(s)$, the corresponding denominator from a minimal factorisation of $G_F(s)$ is $(D(s)+N(s)F)$. Hence it is clear that in general $G(s)$ and $G_F(s)$ do not have the same set of finite and infinite poles. However, although the individual poles are not invariant in this situation, collectively they do display an invariant feature..

(2.15):Theorem: The total number of poles, both finite and infinite of the open loop transfer function matrix $G(s)$ is invariant under constant output feedback although the number of finite poles may change.

Proof: By theorem (2.3.14) the McMillan degree $\delta(G)$ represents the total number of poles, both finite and infinite of $G(s)$. Rosenbrock and Hayton (1974) have shown that $\delta(G)$ is invariant under output feedback and hence the total number of poles is also invariant. However, the total number of finite poles is, by theorem (2.3.10) represented by the least order of G(s), denoted $\nu(g)$ and this is not invariant, as is well-known. Thus the number of finite poles is not invariant.

The next example which is a continuation of example (3.4.10) demonstrates the results of this theorem.

<u>(2.16):Example</u>:Consider again

$$G(s) = \begin{bmatrix} \dfrac{1}{(s-1)(s-2)} & \dfrac{s}{s-1} \\ \dfrac{-s}{s-2} & 1-2s \end{bmatrix}$$

which has a minimal factorisation

$$G(s) = \begin{bmatrix} (s-1)(s-2) & 0 \\ 2(s-1) & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s(s-2) \\ -1 & 1 \end{bmatrix}.$$

As was seen in example (3.4.10) G(s) has one finite pole at s=1, one finite pole at s=2 and one infinite pole. i.e. $\nu(G) = 2$ and $\delta(G) = 3$.

Now consider the closed loop transfer function matrix $G_F(s)$ obtained when feedback

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is applied. Then, from theorem (2.4),

$$G_F(s) = \begin{bmatrix} (s-1)(s-2)+1 & s(s-2) \\ 2s-3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s(s-2) \\ -1 & 1 \end{bmatrix}$$

is a minimal factorisation of $G_F(s)$ and $G_F(s)$ has three finite poles of degree 1 at s=0.63, s=1 and s=2.37, i.e.

$\nu(G_F)=3$.

As in example (3.4.10), and using the notation of that example, $\Lambda(w) = \begin{bmatrix} w^2 & 0 \\ 0 & w \end{bmatrix}$.

Thus the infinite poles of $G_F(s)$ are the zeros at w=0 of

$$\begin{bmatrix} w^2 & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} (\frac{1}{w})-1)(\frac{1}{w}-2)+1 & \frac{1}{w}(\frac{1}{w}-2) \\ \frac{2}{w}-3 & 1 \end{bmatrix} = \begin{bmatrix} 1-3w-2w^2 & 1-2w \\ 2-3w & w \end{bmatrix}.$$

However this matrix has no zeros at w=0 and so $G_F(s)$ has no infinite poles.

i.e. $\nu(G_F) = \delta(G_F)=3$.

Hence it is seen that

$$\delta(G) = \delta(G_F)$$

but

$$\nu(G) \neq \nu(G_F).$$

Thus the application of constant output feedback does not create or destroy poles in any sense but it merely alters their position in the complex s-plane and if $\nu(G)$, the number of finite poles, changes there is a compensating change in the number of poles at infinity. An immediate question thus arises concerning the possibility of using constant output feedback to place all the poles of a given transfer function matrix at finite locations. When all the poles of $G_F(s)$ are situated at finite locations for some F

$$\nu(G_F) = \delta(G_F)$$

which is the precise requirement that $G_F(s)$ be proper.

(Corollary (2.3.15)). Thus the question raised is equivalent to determining when the feedback system transfer function matrix $G_F(s)$ is proper. The answer to this is as follows.

(2.17): Theorem:

(i) $G_F(s)$ is strictly proper if and only if $G(s)$ is strictly proper.

(ii) If $G(s)$ is not strictly proper then $G_F(s)$ is almost always proper.

Proof: The result (i) was established by Rosenbrock and Pugh (1974) while (ii) is due to Anderson and Scott (1976).

As a consequence of this theorem it turns out that almost any feedback matrix $F$ will position all the poles of $G_F(s)$ at finite locations. It is therefore natural to seek the precise conditions under which $G_F(s)$ does possess infinite poles. Clearly, as can be seen from (2.17(i)) this question only arises when $G(s)$ is not strictly proper. The following theorem answers this question.

(2.18): Theorem: If

$$G(s) = G_S(s)+D(s) \tag{2.19}$$

where $G_S(s)$ is strictly proper and $D(s)$ ($\neq 0$) is polynomial then $G_F(s)$ is proper if and only if

$$\delta(\left|I_m+D(s)F\right|) = \delta(D(s)) \tag{2.20}$$

Proof: Let

$$P(s) = \begin{bmatrix} sI_\nu-A & B \\ -C & D(s) \end{bmatrix} \tag{2.21}$$

be a least order realisation of $G(s)$ where $\nu = \nu(G(s))$. Then it is well known (Rosenbrock (1970)) that $D(s)$ as occurs in (2.21) is identical to $D(s)$ as defined by (2.19) and that

$$\delta(G(s)) = \nu+\delta(D(s)). \tag{2.22}$$

Now a least order representation of $G_F(s)$ is

$$P_F(s) = \left[\begin{array}{ccc:c} sI_\nu-A & B & O & O \\ -C & D(s) & -I_m & O \\ O & I_\ell & F & -I_\ell \\ \hdashline O & O & I_m & O \end{array}\right]$$

By constant transformations of strict system equivalence $P_F(s)$ can be reduced, after trivial reduction, to the form

$$P_F'(s) = \left[\begin{array}{cc:c} sI_\nu-A & BF & B \\ -C & I_m+D(s)F & D \\ \hdashline O & -I_m & O \end{array}\right] . \tag{2.23}$$

Now such transformations do not destroy the least order nature of the system matrix nor the associated transfer function matrix $G_F(s)$. Thus (2.23) has the same least order as $P_F(s)$ and so if $\nu_F = \nu(G_F(s))$ then

$$\nu_F = \delta\left[\begin{vmatrix} sI_\nu-A & BF \\ -C & I_m+D(s)F \end{vmatrix}\right] . \tag{2.24}$$

The determinant

$$\begin{vmatrix} sI_\nu-A & BF \\ -C & I_m+D(s)F \end{vmatrix} \tag{2.25}$$

can be expanded by the first $\nu$ rows using the Laplace expansion. Clearly the highest degree for determinants generated from the first $\nu$ rows of (2.25) is $\nu$. Further, the highest degree among minors of all orders of

$$(-C,\ I_m+D(s)F) \tag{2.26}$$

is its McMillan degree, and

$$\delta(-C,\ I_m+D(s)F) = \delta(I_m+D(s)F).$$

Hence $\delta(D(s))$ is an upper bound for the degree of minors of all orders of (2.26).

Now if $\delta(I_m+D(s)F) = \delta(D(s))$ then the above Laplace expansion of (2.25) will contain a term of degree $\nu+\delta(D(s))$. From the form of (2.25) it follows that this is the only term that can possess this degree and in fact all other terms in the Laplace expansion will have degree strictly less than $\nu+\delta(D(s))$.

Hence (2.25) has degree $\nu+\delta(D(s))$ if and only if (2.20) holds. Now if this condition is satisfied, since $P_F(s)$ has least order,

$$\nu_F = \nu+\delta(D(s))$$

$$= \delta(G)$$

from equation (2.22). However (Rosenbrock and Hayton (1974)),

$$\delta(G) = \delta(G_F)$$

and thus

$$\nu_F = \delta(G_F).$$

i.e. $G_F(s)$ is proper.

On the other hand, if $G_F(s)$ is proper then the above argument may be reversed to show that

$$\nu_F = \nu+\delta(D(s)). \tag{2.27}$$

Since (2.27) holds if and only if (2.20) holds it follows that $G_F(s)$ is proper if and only if

$$\delta(\,|I_m+D(s)F|\,) = \delta(D(s))$$

as required.

(2.28):Corollary:The feedback system transfer function matrix has no infinite poles if and only if

$$\delta(|I_m+D(s)F|) = \delta(D(s)).$$

Proof:This follows immediately from the above theorem and corollary (2.2.13).

In the case when G(s) is proper and thus $D(s) \equiv D$, a constant matrix, then the conditions of theorem (2.18) can be further simplified as the next corollary shows.

(2.29):Corollary:If G(s) is proper and

$$G(s) = G_S(s)+D$$

where $G_S(s)$ is strictly proper and D is constant, then $G_F(s)$ is proper if and only if

$$|I_m+DF| = 0.$$

Proof:Since G(s) is proper

$$\begin{aligned} \nu &= \delta(G(s)) \\ &= \delta(G_F(s)). \end{aligned} \qquad (2.30)$$

Also, as in theorem (2.18),

$$\nu_F = \delta\left[\begin{vmatrix} sI_\nu-A & BF \\ -C & I_m+DF \end{vmatrix}\right] . \qquad (2.31)$$

By performing a Laplace expansion of the determinant in (2.31) by the first $\nu$ rows it is seen that

$$\nu_F \leqslant \nu$$

with equality holding if and only if

$$|I_m+DF| \neq 0. \qquad (2.32)$$

It thus follows from (2.30) that

$$\nu_F = \delta(G_F(s))$$

if and only if (2.32) obtains as required.

As a consequence of theorem (2.18) and its corollaries it is clear that if either

$$\delta(|I_m+D(s)F|)\neq\delta(D(s)) \tag{2.33}$$

or, in the case of proper G(s) if

$$|I_m+DF|\neq 0 \tag{2.34}$$

then $\nu_F<\nu$ and hence $G_F(s)$ is non-proper, having $\nu_F$ finite poles and $\nu-\nu_F$ infinite poles. Thus the conditions (2.33) and (2.34) are necessary and sufficient conditions for a particular feedback matrix F to place certain of the poles of a given G(s) at infinity as the following examples illustrate.

<u>(2.35):Example</u>:Continuing example (2.16) let

$$G(s) = \begin{bmatrix} \dfrac{1}{(s-1)(s-2)} & \dfrac{s}{s-1} \\[2ex] \dfrac{-s}{s-2} & 1-2s \end{bmatrix}$$

and

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

If

$$G(s) = G_S(s)+D(s)$$

as in equation (2.19) then

$$D(s) = \begin{bmatrix} 0 & 1 \\ -1 & 1-2s \end{bmatrix}$$

and

$$(I+D(s)F) = \begin{bmatrix} 1 & 1 \\ -1 & 2-2s \end{bmatrix} .$$

Thus $\delta(D(s)) = 1$

$$= \delta(|I+D(s)F|)$$

so that theorem (2.18) predicts that $G_F(s)$ is proper, i.e. $G_F(s)$ has no infinite poles. This was shown to be the case in example (2.16).

More generally if

$$F = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$$

then $$|I+D(s)F| = \begin{vmatrix} 1+f_3 & f_4 \\ -f_1+f_3(1-2s) & 1-f_2+f_4(1-2s) \end{vmatrix}$$

$$= -2sf_4 + \text{constant terms}.$$

Now $\delta(D(s))=1$ and $\delta(|I+D(s)F|)=1$ if and only if $f_4 \neq 0$. Thus, by theorem (2.18) the feedback system is non-proper if and only if $f_4=0$.

The final example in this section illustrates corollary (2.29).

(2.36):Example:Consider the system described by the proper transfer function matrix

$$G(s) = \begin{bmatrix} 1 & \frac{1}{s-1} \\ \frac{1}{s-1} & \frac{1}{s} \end{bmatrix}$$

Thus $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Now apply feedback

$$F = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$$

so that

$$|I+DF| = \begin{vmatrix} 1+f_1 & f_2 \\ 0 & 1 \end{vmatrix}$$

$$= 1+f_1$$

Thus, by corollary (2.29), the feedback system is proper if and only if $f_1 \neq -1$. (2.37)

In fact, if

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$G_F(s) = \begin{bmatrix} \dfrac{(s+1)(s-1)^2-s}{2(s+1)(s-1)^2-s} & \dfrac{s(s-1)}{2(s+1)(s-1)^2-s} \\ \dfrac{s(s-1)}{2(s+1)(s-1)^2-s} & \dfrac{-s+2(s-1)^2}{2(s+1)(s-1)^2-s} \end{bmatrix} \quad (2.38)$$

which is proper whereas if

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$G_F(s) = \begin{bmatrix} \dfrac{(s+1)(s-1)^2}{s} - 1 & (s-1) \\ (s-1) & 1 \end{bmatrix}$$

which is non-proper as is predicted by equation (2.37).

## Section (5.3): Multivariable Lags

A wellknown type of transfer function in single input/single output linear system theory is the kth order lag which has the form

$$g(s) = \frac{1}{a_0 s^k + a_1 s^{k-1} + \ldots + a_{k-1}s + a_k} \qquad (3.1)$$

where $a_o \neq 0$. Note that $\frac{1}{g(s)}$ is a polynomial function and clearly g(s) has

(i) no finite zeros and k infinite zeros

(ii) k finite poles and no infinite poles.

The multivariable theory equivalent of the scalar kth order lag g(s) is the multivariable kth order lag which Owens (1978 p.148) defines as follows.

(3.2): Definition: An m-input/m-output strictly proper system described by the mxm transfer function matrix G(s) is said to be a multivariable kth order lag if and only if $|G(s)| \not\equiv 0$ and

$$G^{-1}(s) = A_0 s^k + A_1 s^{k-1} + \ldots + A_{k-1}s + A_k \qquad (3.3)$$

where

$$|A_0| \neq 0$$

and $A_0, A_1, \ldots, A_k$ are constant matrices.

The results developed in Chapter 2 allow the following observations to be made about kth order multivariable lags, all of which are analogous to the results described above concerning the scalar kth order lag g(s).

(3.4): Theorem: If G(s) is a kth order multivariable lag then G(s) has no infinite poles and no finite zeros.

Proof: Since $G(s)$ is strictly proper, by corollary (2.2.13), it has no infinite poles. Also $G^{-1}(s)$ is polynomial and consequently, by corollary (2.2.11), it has no finite poles. But, by theorem (2.3.19) the finite zeros of $G(s)$ are the finite poles of $G^{-1}(s)$, since $G(s)$ is square and invertible and hence $G(s)$ has no finite zeros.

Owens (ibid p.150) has also shown that $G(s)$ has no finite zeros, using a different approach. In fact, rather more than this can be said and the actual number of poles and zeros, both finite and infinite, of $G(s)$ can be determined as the following theorem shows.

(3.5): Theorem: If $G(s)$ is an $m \times m$ $k$th order multivariable lag then $G(s)$ has $km$ infinite zeros and $km$ finite poles.

Proof: Since $G(s)$ has no infinite poles, by theorem (2.3.19) the full rank polynomial matrix $G^{-1}(s)$ has no infinite zeros and therefore, by theorem (3.1.2) $G^{-1}(s)$ has a high order minor of degree equal to its McMillan degree $\delta(G^{-1}(s))$. In fact, since $G^{-1}(s)$ is square, this high order minor is equal to $|G^{-1}(s)|$, which has degree $km$ because $|A_0| \neq 0$. Thus

$$\begin{aligned}\delta(G^{-1}(s)) &= \delta|G^{-1}(s)| \\ &= km.\end{aligned}$$

But the total number of poles of $G^{-1}(s)$ is equal to its McMillan degree, by theorem (2.3.14), and since $G^{-1}(s)$ is polynomial it has no finite poles. Clearly therefore $G^{-1}(s)$ has $km$ infinite poles and, by theorem (2.3.19) $G(s)$ has $km$ infinite zeros. Since, by theorem (2.3.20), $G(s)$ has the same number of poles as zeros it follows from theorem (3.4) that G(s) has km finite poles.

Some particulary interesting results occur when constant output feedback is applied to such a system. The most important result is the following.

(3.6):Theorem: When constant output feedback is applied to a system with square and invertible transfer function matrix G(s), the feedback system is a kth order multivariable lag if and only if the original system is also a kth order multivariable lag.

Proof: Assume firstly that the system described by G(s) is a kth order multivariable lag. i.e. G(s) is strictly proper and

$$G^{-1}(s) = A_0 s^k + A_1 s^{k-1} \ldots + A_{k-1}s + A_k \tag{3.7}$$

where $|A_0| \neq 0$ and $A_0, A_1, \ldots, A_k$ are constant matrices. Applying constant output feedback F the transfer function matrix becomes (providing as will be assumed that $|I_m + G(s)F| \neq 0$)

$$\begin{aligned} G_F(s) &= (I_m + G(s)F)^{-1} G(s) \\ &= (G^{-1}(s) + F)^{-1} \end{aligned} \tag{3.8}$$

since G(s) is square and invertible. Substituting for $G^{-1}(s)$ from (3.7) in (3.8) gives

$$G_F^{-1}(s) = A_0 s^k + A_1 s^{k-1} + \ldots + A_{k-1}s + A_k + F \tag{3.9}$$

Hence $G_F^{-1}(s)$ is also a polynomial matrix with $|A_0| \neq 0$. Also, by theorem (2.17) since G(s) is strictly proper $G_F(s)$ is also strictly proper. Consequently the feedback system is also a kth order multivariable lag.

Conversely assume that the feedback system is a kth order multivariable lag. Thus $G_F(s)$ is square and invertible and

$$G_F^{-1}(s) = ((I_m+G(s)F)^{-1}G(s))^{-1}$$

Now since $G_F(s)$ is square and non-singular it has rank m. But

$$G_F(s) = (I_m+G(s)F)^{-1}G(s)$$

and thus the matrices $(I_m+G(s)F)^{-1}$ and $G(s)$ also have rank m. Hence $G(s)$ is also square and invertible so that

$$\begin{aligned} G_F^{-1}(s) &= G^{-1}(s)(I_m+G(s)F) \\ &= G^{-1}(s)+F \\ &= B_0s^k+B_1s^{k-1}+ \;\ldots\; +B_{k-1}s+B_k \qquad (3.10) \end{aligned}$$

where $|B_0| \neq 0$. Rearranging (3.10) gives

$$G^{-1}(s) = B_0s^k+B_1s^{k-1}+ \;\ldots\; + B_{k-1}s+B_k-F \qquad (3.11)$$

which is clearly a polynomial of degree k with $|B_0| \neq 0$. Also, by theorem (2.17) since $G_F(s)$ is strictly proper $G(s)$ is strictly proper. ie The system without feedback is a kth order multivariable lag.

<u>(3.12):Corollary</u>: Given a kth order multivariable lag system A with transfer function matrix $G_A(s)$ such that

$$G_A^{-1}(s) = A_0s^k+A_1s^{k-1}+ \;\ldots\; +A_{k-1}s+A_k$$

a feedback system with transfer function matrix $G_B(s)$ given by $G_B^{-1}(s) = B_0s^k+B_1s^{k-1}+ \;\ldots\; +B_{k-1}s+B_k$

can be obtained by applying constant output feedback to the system A if and only if

$$A_i = B_i \text{ for } i = 0,1,2,\ldots k-1.$$

Proof: This is clear from a comparison of equations (3.7) and (3.9).

(3.13):Corollary: The system obtained by applying constant output feedback to a kth order multivariable lag has

(i) no finite zeros and km infinite zeros

(ii) km finite poles and no infinite poles.

Proof: From theorem (3.6) the feedback system is a kth order multivariable lag and hence the corollary follows immediately from theorems (3.4) and (3.5).

## Section (5.4) : Multivariable Root Locus Theory

Root locus theory for single-input, single-output feedback systems is concerned with the effect on the poles of the closed-loop transfer function of varying the gain k of the feedback transfer function when output feedback is applied as in figure (4.1).

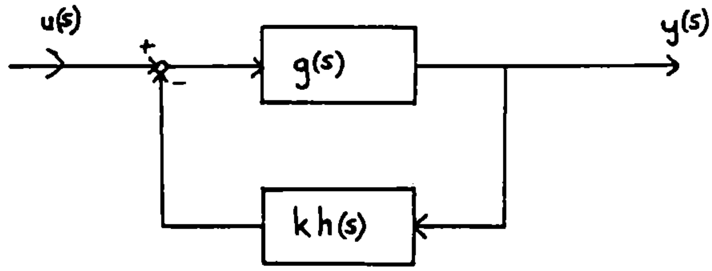


Figure (4.1)

The closed loop system transfer function is of course given by

$$g_f(s) = \frac{g(s)}{1+kg(s)h(s)} .$$

Suppose that

$$g(s) = \frac{n(s)}{d(s)}$$

where n(s) and d(s) have no common factors so that the $\delta_z$ zeros of n(s) are the zeros of g(s) and the $\delta_p$ zeros of d(s) are the poles of g(s). g(s) is assumed to be proper and hence g(s) has $\delta_p - \delta_z$ infinite zeros and no infinite poles. In the simplest case h(s) is taken to be unity and consequently

$$g_f(s) = \frac{n(s)}{d(s)+kn(s)}$$

so that the poles of $g_f(s)$ are the zeros of

$$p_f(s) = d(s) + kn(s). \tag{4.2}$$

The locus of the poles of $g_f(s)$ in the complex plane for varying k can now be drawn. Clearly if k=0 then $p_f(s)=d(s)$ and so the $\delta_p$ branches of the root locus plot begin at the poles of g(s). As $k \to \infty$ the poles of $g_f(s)$ approach the zeros of g(s) and hence the branches of the root locus terminate at the $\delta_z$ finite zeros and $\delta_p - \delta_z$ infinite zeros of g(s). This is illustrated by figure (4.3) which shows a typical root locus diagram for

$$g(s) = \frac{s+a}{(s+b)(s+c)(s+d)}$$

where $d > c > a > b$

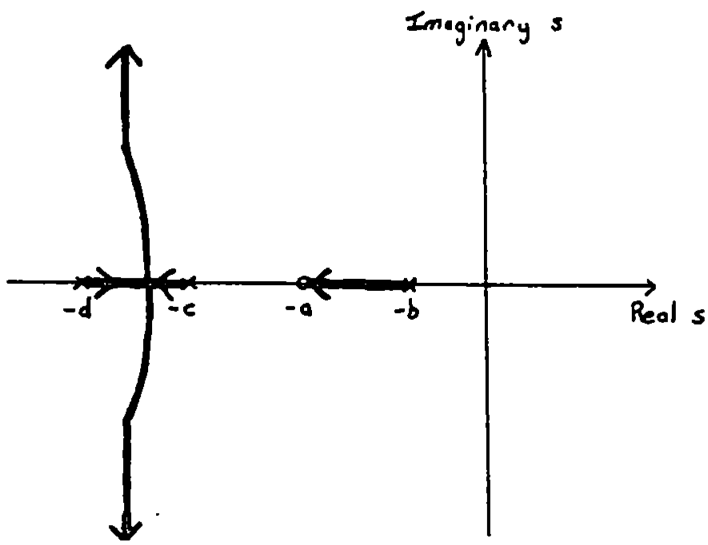


Figure (4.3)

The three branches of the root locus begin at the three finite poles of $g(s)$ and terminate one at the finite zero $s = -a$ and the other two at infinity since $g(s)$ has two infinite zeros.

The root locus technique is a well-established tool for the analysis and design of scalar systems and of course the results described thus far in this section form only a very small part of the theory. One field of current research interest is the generalisation of scalar root locus theory to multivariable systems. (See for example Owens (1978), MacFarlane & Postlethwaite (1979), Kouvaritakis and Shaked (1976)). In line with the main theme of this thesis which is the generalisation of results in the theory of scalar rational functions to rational matrices it is the aim of this section to make use of some of the results obtained thus far in order to generalise some of the above results for scalar root locus theory to multivariable systems.

Clearly it is not feasible to consider the effect on the feedback transfer function matrix poles of varying the gain of all the independent elements of the feedback matrix F and so in multivariable root locus theory attention is usually restricted to the case of output feedback of the form

$$F = pK$$

where K is a constant matrix and p is a parameter known as the overall gain. In this section consideration will be further restricted to systems with square (mxm) invertible transfer function matrix $G(s)$ and feedback matrix F given by

$$F = pI_m. \tag{4.4}$$

In this way the closest possible analogy to scalar systems is obtained.

In the case of scalar $g(s)$ it is obvious that $g(s)$ has the same number of poles (finite and infinite) as zeros. In the multivariable case this is usually taken as self-evident although the infinite zeros are never specifically defined in terms of the rational matrices involved but are taken as the terminal destinations of the multivariable root loci as $p \to \infty$. It is the intention of this section to define infinite zeros by reference only to the rational matrices concerned. In this manner corollary (2.3.20) confirmed that a square invertible matrix $G(s)$ does indeed have the same number of poles as zeros. This situation does not obtain if $G(s)$ is non-invertible nor if it is non-square.

Now let

$$G(s) = D^{-1}(s) N(s) \tag{4.5}$$

be a minimal factorisation of $G(s)$ and let the row degrees of $(D(s),N(s))$ be denoted by $\delta_i$ $(i = 1,2,\ldots,m)$. If constant output feedback $F$ as in equation (4.4) is applied then the feedback system transfer function matrix is given by

$$\begin{aligned} G(s,p) &\equiv G_F(s) \\ &= (I_m + G(s).pI_m)^{-1} G(s) \\ &= (I_m + pG(s))^{-1} G(s) \end{aligned}$$

and so, by theorem (2.4),

$$G(s,p) = (D(s) + N(s)p)^{-1} N(s)$$

is a minimal factorisation of G(s,p) for fixed p. Then

$$G(s,p) = (pI_m(\frac{1}{p}I_m.D(s) + N(s)))^{-1} N(s)$$

$$= (\frac{1}{p}D(s) + N(s))^{-1} \frac{1}{p}N(s) \tag{4.6}$$

is also a minimal factorisation of G(s,p) for fixed p ($\neq 0$). Thus, by theorem (2.13), the following theorem may be stated.

<u>(4.7):Theorem</u>: The finite poles of G(s,p) (for $p \neq 0$) are the finite zeros of

$$\frac{1}{p}D(s) + N(s) \tag{4.8}$$

and the infinite poles are the zeros at w=0 of

$$\Lambda(w) (\frac{1}{p}D(\frac{1}{w}) + N(\frac{1}{w})) \tag{4.9}$$

where

$$\Lambda(w) = \text{diag } (w^{\delta_1}, w^{\delta_2}, \ldots, w^{\delta_m}).$$

As was shown above in scalar root locus theory as the gain $k \to \infty$ the poles of the closed loop transfer function approach the zeros of the open loop transfer function, i.e. the branches of the root locus terminate at the open loop zeros. Letting $p \to \infty$ in the results of theorem (4.7) yields the following theorem which shows that analogous results hold for the multivariable root locus.

<u>(4.10):Theorem</u>: As the gain $p \to \infty$ the various branches of the multivariable root locus terminate at the finite and infinite zeros of the openloop transfer function matrix.

<u>Proof</u>: Letting $p \to \infty$ in (4.8) and (4.9) shows that the finite poles of $G(s,\infty)$ are the finite zeros of N(s) whilst

the infinite poles of $G(s,\infty)$ are the zeros at $w=0$ of the polynomial matrix $\Lambda(w)N(\frac{1}{w})$. However, $N(s)$ is a numerator from a minimal factorisation of $G(s)$ and $\Lambda(w)$ is the corresponding diagonal matrix for this factorisation. Thus, by theorem (3.4.1), the result follows immediately.

From this theorem it can be seen that the infinite zeros that are usually referred to in multivariable root locus theory can be clearly defined as the infinite zeros of the open loop transfer function matrix and without any reference being made to the multivariable root locus.

It is usual in multivariable root locus theory to consider only the case of strictly proper $G(s)$. Hence, by theorem (2.17), $G(s,p)$ is strictly proper for all $p$. i.e. $G(s,p)$ has no infinite poles and thus no branches of the root locus move off to infinity for finite values of $p$. If however $G(s)$ is allowed to be proper or non-proper, theorem (2.18) and corollary (2.29) predict finite values of $p$ for which this situation arises.

<u>(4.11):Theorem</u>: If $G(s)$ is non-proper and

$$G(s) = G_s(s) + D(s)$$

where $G_s(s)$ is strictly proper and $D(s)$ is polynomial then the values of the overall gain $p$ for which certain branches of the root locus become unbounded are given by those values of $p$ for which

$$\delta(|I_m + pD(s)|) = \delta(D(s)) \tag{4.12}$$

Proof: This follows immediately from theorem (2.18) substituting $F = pI_m$.

(4.13):Corollary: If G(s) is proper and

$$G(s) = G_s(s) + D$$

where D is a constant matrix then certain branches of the root locus become unbounded when the overall gain p satisfies the equation

$$|I_m + pD| = 0.$$

Proof: This is immediate from corollary (2.29).

Thus the values of the gain parameter p that produce unbounded root locus branches are precisely $\frac{-1}{\lambda}$ where $\lambda$ is an eigenvalue of D. Special consideration is required of the zero eigenvalues of D which correspond to the case $p = \infty$ and is not attempted here. The following example which is a continuation of example (2.36) illustrates corollary (4.13).

(4.14):Example: If

$$G(s) = \begin{bmatrix} 1 & \frac{1}{s-1} \\ \frac{1}{s-1} & \frac{1}{s} \end{bmatrix}$$

then

$$G_s(s) = \begin{bmatrix} 0 & \frac{1}{s-1} \\ \frac{1}{s-1} & \frac{1}{s} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now D has one non-zero eigenvalue at 1 and hence $p = -1$ is a value of the gain parameter p at which the branches of the root locus become unbounded.

<u>Section (5.5)</u> : <u>State Feedback and The Theory of Decoupling</u>

One field of current research in linear multivariable control theory is the theory of decoupling. (See for example Wolovich (1974), Vardulakis and Stoyle (1979), Vardulakis (1980)). This is the study of the application of state-feedback to a system in state-space form in order to obtain a system with diagonal transfer function matrix in which the different states are said to be "decoupled" from each other. Whilst it is not the intention in this thesis to discuss decoupling other than very briefly a number of interesting results relating to the effect of state feedback and the problem of decoupling which follow simply from the results derived thus far will be presented.

Consider a linear multivariable system with no finite decoupling zeros described by the equations

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

where the constant matrices A, B and C have dimensions nxn, nxm and mxn respectively and B and C have full rank $m \leq n$. This system gives rise to a square, strictly proper transfer function matrix

$$G(s) = C(sI-A)^{-1}B \tag{5.1}$$

and an (n+m)x(n+m) system matrix in state-space form

$$P(s) = \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \tag{5.2}$$

State-feedback may be applied according to the equation

$$u = -Fx + Gv$$

where F and G are respectively mxn and mxm constant matrices and $|G| \neq 0$. The feedback system then has transfer function matrix

$$G_F(s) = C(sI-A-BF)^{-1}G \tag{5.3}$$

and system matrix

$$P_F(s) = \begin{bmatrix} sI-A-BF & BG \\ -C & 0 \end{bmatrix}. \tag{5.4}$$

The following result can be obtained immediately from corollary (4.1.29) since both P(s) and $P_F(s)$ are in state-space form.

<u>(5.5):Theorem</u>: Neither P(s) nor $P_F(s)$ has any infinite decoupling zeros.

The next result which is an additional corollary to theorem (4.1.28) describes the simple relationship between the infinite system zeros as described by definition (4.1.20) and the infinite transfer function matrix zeros of any system in state-space form.

<u>(5.6):Corollary</u>: The set of infinite system zeros of a system in state-space form is equal to the set of infinite zeros of the transfer function matrix.

<u>Proof</u>: This follows immediately from equation (4.1.23(b)) since from corollary (4.1.29) any system in state-space form has no infinite decoupling zeros.

In chapter 4 a rather complicated definition, due to Ferreira(1980), of infinite system zeros was given. However, in the case of systems with proper transfer function matrices and system matrices in state-space form this definition can be replaced by the much simpler

(5.7):Definition: The infinite system zeros of a system with a system matrix in state-space form and a proper transfer function matrix are the infinite zeros of the system matrix.

(5.8):Theorem: Definitions (4.1.20) and (5.7) define the same set of zeros.

Proof: Definition (5.7) refers to system matrices of the type

$$P(s) = \begin{bmatrix} sI_n-A & B \\ -C & D \end{bmatrix}. \tag{5.9}$$

Now

$$P(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w}I_n-A & B \\ -C & D \end{bmatrix}$$

$$= \begin{bmatrix} wI_n & 0 \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} I_n-Aw & Bw \\ -C & D \end{bmatrix}.$$

This is a relatively left prime factorisation of $P(\frac{1}{w})$ and thus the infinite zeros of P(s) are the zeros at w=0 of

$$\begin{bmatrix} I_n-Aw & Bw \\ -C & D \end{bmatrix}. \tag{5.10}$$

The normalised system matrix is given by

$$P_N(s) = \left[\begin{array}{cccc} sI_n-A & B & 0 & 0 \\ -C & D & -I_m & 0 \\ 0 & I_\ell & 0 & -I_\ell \\ \hline 0 & 0 & I_m & 0 \end{array}\right]$$

so that

$$P_N(\tfrac{1}{w}) = \left[\begin{array}{ccc:c} \frac{1}{w}I_n-A & B & 0 & 0 \\ -C & D & -I_m & 0 \\ 0 & I_\ell & 0 & -I_\ell \\ \hline 0 & 0 & I_m & 0 \end{array}\right]$$

$$= \begin{bmatrix} wI_n & 0 \\ 0 & I_{2m+\ell} \end{bmatrix}^{-1} \left[\begin{array}{ccc:c} I_n-Aw & Bw & 0 & 0 \\ -C & D & -I_m & 0 \\ 0 & I_\ell & 0 & -I_\ell \\ \hline 0 & 0 & I_m & 0 \end{array}\right]$$

$$= D_2^{-1}(w)N_2(w).$$

This is a relatively left prime factorisation of $P_N(\frac{1}{w})$ and hence $N_2(w)$ is a numerator of $P_N(\frac{1}{w})$. Following definition (4.1.20) the minors of $N_2(w)$ which contain the first $n+m+\ell$ rows and columns must be examined. These minors are either zero or equal to the minors of

$$\begin{bmatrix} I_n-Aw & Bw \\ -C & D \end{bmatrix}. \tag{5.11}$$

Clearly the greatest common divisor of all the high order bordered minors of $N_2(w)$, $d(w)$, is equal to the product of the terms in the Smith form of (5.11) and hence the zeros at w=0 of d(w) are the zeros at w=0 of (5.10) which is a numerator of $P(\frac{1}{w})$ and thus the result follows.

The system matrices under consideration in this section described by (5.2) and (5.4) are special cases of the system matrix (5.9) with D=0 and hence, when the zeros of the system are being investigated, definition (5.7) will be utilised instead of the more complicated definition (4.1.20). In view of this result and corollary (5.6) the following corollary can be stated.

<u>(5.12):Corollary</u>: The infinite zeros of a strictly proper transfer function matrix $G(s) = C(sI_n-A)^{-1}B$ are given by the infinite zeros of the associated system matrix

$$P(s) = \begin{bmatrix} sI_n-A & B \\ -C & 0 \end{bmatrix} .$$

The next theorem and its corollary formed Proposition 3 in Vardulakis (1980) but they were not rigorously proved by him and the complete proofs depend on corollary (5.6) and theorem (5.8).

<u>(5.13):Theorem</u>: Infinite system zeros are invariant under state feedback.

<u>Proof</u>: Following definition (5.7) the infinite zeros of the original system are the zeros at w=0 of

$$P(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w}I_n - A & B \\ -C & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w}I_n & 0 \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} I_n - Aw & Bw \\ -C & 0 \end{bmatrix}.$$

This is a relatively left prime factorisation and hence the infinite zeros of the system are the zeros at w=0 of the numerator

$$N(w) = \begin{bmatrix} I_n - Aw & Bw \\ -C & 0 \end{bmatrix}.$$

Similarly the infinite zeros of the system with state feedback are the zeros at w=0 of

$$P_F(\tfrac{1}{w}) = \begin{bmatrix} \frac{1}{w}I_n - A - BF & BG \\ -C & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w}I_n & 0 \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} I_n - Aw - BFw & BGw \\ -C & 0 \end{bmatrix}.$$

This is also a relatively left prime factorisation and thus the infinite zeros of $P_F(s)$ are the zeros at w=0 of

$$N_F(w) = \begin{bmatrix} I_n - Aw - BFw & BGw \\ -C & 0 \end{bmatrix}.$$

But

$$N_F(w) = N(w) \begin{bmatrix} I_n & 0 \\ -F & G \end{bmatrix}$$

and $|G| \neq 0$.

Hence $N_F(w)$ and $N(w)$ have the same zero structure and in particular they have the same zeros at w=0. Thus P(s) and $P_F(s)$ have the same infinite zeros and consequently, by definition (5.7), the infinite system zeros of the original system and the system with statefeedback are identical.

(5.14):Corollary: The infinite zeros of the transfer function matrix are invariant under state feedback.

Proof: This follows immediately from corollary (5.6) and theorem (5.13).

Vardulakis (1980) defines a set of integers known as the "decoupling invariants" of a system. These integers, denoted $f_i$ $(i=1,2,\ldots,m)$ are characteristics of the original system without feedback and can be calculated using various constructions which Vardulakis outlines. The conditions for a system such as is described by (5.2) to be decouplable will not be detailed here but the following results, given by Vardulakis are worth mentioning as they follow directly from the previous results in this section.

(5.15):Theorem: If the system is decouplable then the number of finite zeros q of the system and also (since the system is assumed to have no finite decoupling zeros) of G(s) is given by

$$q = n - m - \sum_{i=1}^{m} f_i .$$

Vardulakis shows that a system with transfer function matrix

$$T_{\tilde{F}^*\tilde{G}^*}(s) = \begin{bmatrix} \frac{1}{s^{\xi_1+1}} & & & 0 \\ & \frac{1}{s^{\xi_2+1}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{s^{\xi_m+1}} \end{bmatrix} \qquad (5.16)$$

can be obtained by the addition of state feedback to any decouplable system. It is clear that $T_{\tilde{F}^*\tilde{G}^*}(s)$ has m infinite zeros, each of degree $\xi_i+1$, i=1,2,...m. Hence, by theorem (5.13) and corollary (5.14) it follows that

<u>(5.17):Theorem</u>: If a system is decouplable then G(s) (and also $G_F(s)$ for any state feedback matrices F and G) has m infinite zeros, each infinite zero having degree $n_i=\xi_i+1$, $i=1,2,\ldots,m$.

In the case of non-decouplable systems Vardulakis also shows that G(s) has m infinite zeros but in this case their degrees are determined by a rather long construction which will not be detailed here.

Thus this section has shown that the number of infinite system zeros and infinite transfer function zeros of a linear multivariable system with no finite decoupling zeros and square, non-singular, strictly proper transfer function matrix is invariant under state feedback. It has also been noted that the number of infinite zeros of a decouplable system is directly related to the decoupling invariants of that system.

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## Appendix 1: Linearisation of L(s) considering the matrix as a whole.

Consider any mxn polynomial matrix L(s) and let the highest degree of any element of L(s) be $\ell$ so that L(s) may be expanded as

$$L(s) = A_\ell s^\ell + A_{\ell-1}s^{\ell-1} + \dots + A_1 s + A_o$$

where $A_i$ $(i = 0, 1, \dots, \ell)$ are mxn constant matrices. Define

$$B_o(s) = A_\ell$$

and

$$B_{r+1}(s) = sB_r(s) + A_{\ell-r-1}, \quad r=0,1,\dots,\ell-2.$$

Then L(s) may be linearised according to the equation

$$L(s)F_1(s) = E_1(s)\ (sE^*-A^*)$$

where

$$F_1(s) = (I_n, O_{n,n(\ell-1)})$$

$$E_1(s) = (B_{\ell-1}(s), B_{\ell-2}(s), \dots\dots B_1(s), I_m)$$

and

$$(sE^*-A^*) = \begin{bmatrix} sI_n & -I_n & O & \cdots & \cdots & O \\ O & sI_n & -I_n & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & O \\ O & O & O \cdots O & & sI_n & -I_n \\ A_o & A_1 & A_2 \cdots A_{\ell-3} & & A_{\ell-2} & A_{\ell-1}+sA_\ell \end{bmatrix}$$

Appendix 2: Linearisation of L(s) considering the columns separately.

Consider an mxn polynomial matrix L(s) consisting of columns $L_1(s), L_2(s), \ldots, L_n(s)$ and let the highest degree of any element in $L_i(s)$ be denoted $\delta_i$ $(i = 1,2,\ldots,n)$ so that

$$L_i(s) = L_i^{\delta i} s^{\delta i} + L_i^{\delta i-1} s^{\delta i-1} + \ldots + sL_i^1 + L_i^0$$

where $L_i^k$, $k = 0,1,\ldots,\delta_i$ are constant m vectors.

Now define for $i = 1,2,\ldots,n$

$$B_i^1(s) = L_i^{\delta i-1} + sL_i^{\delta_i}$$

and

$$B_i^{k+1}(s) = sB_i^k(s) + L_i^{\delta i-k-1}, \quad k = 1,2,\ldots,\delta_i-2.$$

If $\delta_i>1$ define

$$F_i(s) = (1, 0_{1,\delta i-1})$$

$$P_i(s) = (L_i^0, L_i^1, \ldots, L_i^{\delta i-2}, B_i^1)$$

and let the $(\delta_i-1)x\delta_i$ polynomial matrix

$$E_i(s) = \begin{bmatrix} s & -1 & 0 & \cdots & \cdots & 0 \\ 0 & s & -1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & s & -1 \end{bmatrix}$$

whereas if $\delta_i \leqslant 1$ define

$$F_i(s) = (1)$$

$$P_i(s) = L_i^0$$

and

$$E_i(s) = (O_{\delta p-1,1})$$

where $\delta p$ is the degree of the next column for which $\delta_i > 1$ .

Then L(s) may be linearised according to the equation $L(s)\, F(s) = B(s)\,(sE' - A')$

where

$$F(s) = \text{diag}\,(F_1(s),\ F_2(s),\ \ldots,\ F_n(s))$$

$$B(s) = (B_1^{\delta_1-1},\ B_1^{\delta_1-2},\ \ldots\ B_1^1 \vdots B_2^{\delta_2-1},\ B_2^{\delta_2-2},\ \ldots\ B_2^1 \vdots \cdots \vdots B_\ell^{\delta_\ell-1},\ B_\ell^{\delta_\ell-2},\ \ldots\ B_\ell^1 \vdots\ I_m)$$

where if $\delta j \leqslant 1$ there are no columns $B_j$ in B(s)

and

$$(sE'-A') = \begin{bmatrix} E_1 & O & O & \cdots & \cdots & \cdots & O \\ O & E_2 & O & & & & O \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & E_{p-1} & E_p & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & O \\ O & \cdots & \cdots & \cdots & \cdots & O & E_\ell \\ P_1 & P_2 & & P_{p-1} & P_p & \cdots & P_\ell \end{bmatrix}$$

where, in order to illustrate the case when for some $j$, $\delta_j \leqslant 1$, then $\delta_{p-1} \leqslant 1$ and $\delta_p > 1$.

# INFINITE POLE AND ZERO CONSIDERATIONS FOR SYSTEM TRANSFER FUNCTION MATRICES

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ABSTRACT

The finite and infinite poles and zeros of a rational matrix are defined in a manner entirely consistent with the scalar case of rational functions. Following a discussion of the relevance of row and column properness in infinite frequency considerations an important structural theorem is indicated. Subsequently this result is exploited in a study of the effect of constant output feedback on the totality of open-loop poles and zeros.

## 1. INTRODUCTION

The finite poles and zeros of a transfer function matrix have been defined and discussed by many authors and summaries may be found in Rosenbrock (1970), Wolovich (1974) and Kailath (1980). The definitions developed in this paper make precise Wolovich's (1974) ideas of numerators and denominators of rational matrices thus enabling poles and zeros to be defined in an entirely analogous manner to that employed in the scalar case of rational functions. The definition of infinite poles and zeros then follows by utilising the standard bilinear transformation of complex variable theory in the manner suggested by McMillan (1952).

In section 3 a discussion is presented on the relevance of row and column properness in infinite frequency considerations and it is found to imply a fundamental structural property of factorisations of rational matrices. An investigation into the effects of constant output feedback on the totality of zeros of a transfer function matrix provides a ready application of this structural property. Subsequently a study is made of the pole locations under output feedback. Some reinterpretations are offered and some very recent results concerning the existence of infinite poles for the feedback transfer function matrix are noted.

## 2. POLES AND ZEROS OF A RATIONAL MATRIX

If g(s) is a scalar transfer function and

$$g(s) = \frac{n(s)}{d(s)} \tag{2.1}$$

where n(s) and d(s) are relatively prime polynomials then the finite zeros (respectively poles) of g(s) are defined as the zeros of any numerator n(s) (resp. denominator d(s)). Such quantities are clearly well-defined since all numerators (resp. denominators) differ only by some constant scalar factor. We wish to define the finite poles and zeros of a rational matrix in a manner that bears the closest possible analogy to this scalar case.

To begin with we define the finite zeros of a polynomial matrix.

<u>Definition 1</u> $s_o \in \mathbb{C}$ is a FINITE ZERO OF DEGREE k of the $m \times \ell$ polynomial matrix D(s) in case $(s-s_o)^k$ is an elementary divisor of D(s). The set of finite zeros of D(s) is the set of all such numbers $s_o$, any zero of degree k being included k times.

This is essentially Rosenbrock's (1974) definition wherein the zeros of D(s) are defined as the zeros of the invariant polynomials of D(s) taken all together. The only difference between these definitions lies in the concept of degree, thus in either case a necessary and sufficient condition for $s_o$ to be a finite zero is that the rank of $D(s_o)$ be less than the normal rank $\rho(D(s))$ of D(s).

To identify the numerators and denominators in the case of a matrix of rational functions G(s) we utilise the well-known decomposition of such a matrix into relatively prime factors

$$\text{i.e.} \qquad G(s) = N_1(s)\, D_1^{-1}(s) = D_2^{-1}(s)\, N_2(s) \tag{2.2}$$

where $N_1(s)$ and $D_1(s)$ are relatively right prime and $D_2(s)$, $N_2(s)$ are relatively left prime. As suggested by Wolovich (1974) we say

Definition 2 Any $m \times \ell$ matrix such as $N_1(s)$ or $N_2(s)$ satisfying (2.2) is called a NUMERATOR of $G(s)$. Any $\ell \times \ell$ polynomial matrix such as $D_1(s)$ or $m \times m$ polynomial matrix such as $D_2(s)$ is called a DENOMINATOR of $G(s)$.

The applicability of this terminology to the problem of defining the poles and zeros of $G(s)$ is not immediately clear, for although all numerators of $G(s)$ are known to be equivalent in the sense of unimodular matrix transformations (Rosenbrock, 1970; Wolovich, 1974) the exact relationship between the denominators is not so readily apparent since in particular different denominators may have different dimensions. Thus to define the poles of $G(s)$ as the zeros of any denominator is unacceptable in such circumstances.

These difficulties however may be resolved due to the work of Fuhrmann (1977) who introduced the following transformation of polynomial matrices.

Definition 3 Let $P(m,\ell)$ be the class of $(r+m)\times(r+\ell)$ polynomial matrices where $\ell, m$ are fixed integers and $r$ ranges over all integers which are greater than $\max(^-m,^-\ell)$. The matrices $P(s)$, $P_1(s) \in P(m,\ell)$ are said to be EXTENDED UNIMODULAR EQUIVALENT in case there exist polynomial matrices $M(s)$, $N(s)$ such that

$$M(s)P(s) = P_1(s)N(s) \tag{2.3}$$

where $P_1(s)$ and $M(s)$ are relatively left prime and $P(s),N(s)$ are relatively right prime.

The terminology of this definition is not Fuhrmann's but that of Pugh and Shelton (1978) and is suggested by certain properties of the proposed transformation. Firstly the transformation (2.3) is one of equivalence on the class $P(m,\ell)$ as Pugh and Shelton (ibid.) using direct matrix methods, and subsequently Kailath (1980) have shown. Secondly anything that can be accomplished by the transformation (2.3) can be achieved using the more usual transformation of unimodular equivalence together with the operation of trivial expansion (Rosenbrock, 1970), and vice versa. This is embodied in the following result of Pugh and Shelton (1978).

Lemma 1 The matrices $P(s)$, $P_1(s) \in P(m,\ell)$ are extended unimodular equivalent if and only if the Smith form of one of these matrices is a trivial expansion of the Smith form of the other.

The value in the present context of the transformation (2.3) is that under it matrices of differing dimensions may be related. However it follows from Lemma 1 that should the considered matrices be of the same dimensions then they are extended unimodular equivalent if and only if they are equivalent in the unimodular sense.

It is the above ideas that enable the complete relationship between the various numerators and denominators to be established (Pugh and Ratcliffe, 1979).

Theorem 1 The numerators of the rational matrix G(s) are unimodular equivalent while all denominators are extended unimodular equivalent.

It follows from this result and Lemma 1 that

Corollary 1 If $s_o$ is a zero of degree k of a numerator (respectively denominator) of G(s) then it is a zero of degree k of every numerator (resp. denominator).

It is this result that makes the following definition meaningful.

Definition 4 $s_o \in \mathbb{C}$ is a FINITE ZERO (respectively, POLE) of degree k of the $m \times \ell$ rational matrix G(s) in case it is a zero of degree k of any numerator (resp. denominator).

It is clear that the sets of poles and zeros determined by this definition coincide with those defined in the more usual way (Rosenbrock, 1974) via the Smith - McMillan form of G(s). The definition of poles and zeros in the manner described above proves useful from a theoretical and technical point of view more than any practical one of determining poles and zeros.

In order to define the terms "infinite pole" and "infinite zero" we make use of the standard technique of complex variable theory and perform the bilinear transformation

$$s = \frac{1}{\omega} \tag{2.4}$$

This transformation takes the point $s = \infty$ to the point $\omega = 0$ and the point $s = 0$ to the point $\omega = \infty$. All other points in the complex s-plane are carried onto finite points in the complex $\omega$-plane in a one-one manner. We thus say

Definition 5 The $m \times \ell$ rational matrix G(s) possesses an INFINITE ZERO (respectively, POLE) of degree k in case $G(\frac{1}{\omega})$ has a finite zero (resp. pole) of precisely that degree at $\omega = 0$.

As a first, and as it turns out, only characterisation of minimal bases from the infinite frequency point of view we have,

Theorem 2 If the $m \times \ell$ polynomial matrix $P(s)$ forms a minimal basis then it possesses no finite and no infinite zeros.

Proof From (i) of Definition 6 it follows that the Smith form of $P(s)$ is $(I_m, 0_{m,\ell-m})$ and hence that $P(s)$ has no finite zeros.

Let,

$$\Lambda(s) = \mathrm{diag}(s^{\delta_1},\dots,s^{\delta_m}) \tag{3.1}$$

then

$$P(\tfrac{1}{\omega}) = \Lambda(\tfrac{1}{\omega})\,\tilde{P}(\omega) = \Lambda(\omega)^{-1}\,\tilde{P}(\omega) \tag{3.2}$$

is a factorisation of $P(\frac{1}{\omega})$ into polynomial matrices where by the properties of $\Lambda(\omega)$ the matrix $\tilde{P}(\omega)$ is such that

$$\tilde{P}(0) = [P]_h \tag{3.3}$$

Consider the matrix

$$(\Lambda(\omega),\ \tilde{P}(\omega)) \tag{3.4}$$

then for $\omega = 0$ we have on using (3.3)

$$(\Lambda(0),\ \tilde{P}(0)) \equiv (0_{m,m},\ [P]_h) \tag{3.5}$$

Since $P(s)$ is a minimal basis it follows from definition 6(ii) that (3.5) has full rank. Further for any finite $\omega(\neq 0) \in \mathbb{C}$ the matrix (3.4) certainly has rank m because of the form $\Lambda(\omega)$. Hence $(\Lambda(\omega),\ \tilde{P}(\omega))$ is a relatively left prime factorisation of $P(\frac{1}{\omega})$. Now

$$\mathrm{RANK}\ \tilde{P}(0) = \mathrm{RANK}\ [P]_h = m$$

and so $\tilde{P}(\omega)$ and consequently $P(\frac{1}{\omega})$, has no zeros at $\omega = 0$. Hence by definition $P(s)$ has no infinite zeros.

example 1 and the matrix

$$\begin{pmatrix} s & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.8}$$

do not possess the same infinite zeros. However

$$P_1(s) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

so that $P_1(s)$ and (3.8) are only a simple constant transformation away from each other. Thus the infinite zeros as defined by Anderson and Bitmead (1978) are not invariant under the most elementary of transformations. This difficulty arises because the construction devised does not demand a relatively prime factorisation of the rational matrix in hand. Thus "infinite zeros", analogous to decoupling zeros, are introduced over and above those defined in this paper. It is these additional entities which give rise to the situation just described and as a consequence represent dynamically uninteresting properties. A necessary and sufficient condition for the absence of infinite zeros, in the sense defined here, has been provided by Pugh and Ratcliffe (1979).

It is seen from the above discussion that from the point of view of the existence of infinite zeros the concept of row properness does not assume great importance. Nevertheless the concept does possess a very interesting implication for relatively prime factorisations of rational matrices as follows.

Theorem 3 Let $G(s)$ be an $m \times \ell$ rational matrix and

$$G(s) = D(s)^{-1} N(s) \tag{3.9}$$

a polynomial factorisation of $G(s)$ in which

$$(D(s), N(s)) \tag{3.10}$$

forms a minimal basis. Let the $i^{th}$ row degree of (3.10) be denoted by $\delta_i$ $(i = 1,2,\ldots,m)$ and define

$$\Lambda(\omega) = \text{diag}(\omega^{\delta_1},\ldots,\omega^{\delta_m}) \tag{3.11}$$

Then

(i) the finite poles of G(s) are the finite zeros of D(s) and the infinite poles are the zeros at $\omega = 0$ of the polynomial matrix

$$\Lambda(\omega) \quad D(\frac{1}{\omega}) \tag{3.12}$$

(ii) the finite zeros of G(s) are the finite zeros of N(s) and the infinite zeros are the zeros at $\omega=0$ of the polynomial matrix

$$\Lambda(\omega) \quad N(\frac{1}{\omega}) \tag{3.13}$$

Proof This follows in a similar way to Theorem 2, the exact details may be found in Pugh and Ratcliffe (1980).

It is thus seen that not only do the factorisations described in the theorem display the finite pole-zero structure of the given rational matrix (as indeed does any prime polynomial factorisation) but they additionally display the infinite pole-zero structure in a particularly simple and readily exploitable manner. We will refer to such factorisations as MINIMAL to distinguish them from the usual relatively prime factorisations.

## 4. SOME EFFECTS OF OUTPUT FEEDBACK

The effect of constant output feedback on the finite poles and zeros of a transfer function matrix G(s) has been widely considered, and in particular it is known (Rosenbrock, 1970) that the finite zero structure of G(s) is unchanged by such action. The finite pole structure of course possesses no such invariant property. It is clearly of interest to know what changes are caused to the infinite zero and infinite pole structure of G(s) when output feedback is applied.

Accordingly suppose that constant output feedback as summarised by the matrix F is applied to the open loop system described by G(s). If $G_F(s)$ denotes the feedback system transfer function matrix then

$$G_F(s) = G(s)\ (I_{\ell}+FG(s))^{-1} = (I_m+G(s)F)^{-1}G(s) \tag{4.1}$$

provided, as will always be assumed, that

$$|I_{\ell}+FG(s)| \equiv |I_m+G(s)F| \neq 0 \tag{4.2}$$

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# On the zeros and poles of a rational matrix

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This paper considers the poles and zeros of a rational matrix, particularly those situated at infinity. Certain results are established which are generalizations of the rational function case, and several connections are made with previous work.

## 1. Introduction

The finite poles and zeros of a rational matrix have been defined by many authors and a summary may be found in Pugh (1977). Recently attention (Rosenbrock 1974 a), Verghese *et al.* 1978, 1979) has been focused on defining the infinite poles and zeros of a rational matrix. Rosenbrock (1970) and McMillan (1952) have also considered the case of infinite poles and zeros by utilizing a bilinear transformation that takes some point, at which there are no finite poles or zeros, to infinity. In this way the number of poles, finite and infinite, of a given rational matrix is seen to be equal to its McMillan degree.

In this paper the main definitions are given in § 2 and the infinite poles and zeros are defined via a standard technique of complex variable theory. This procedure enables several results concerning rational functions to be generalized to rational matrices, and these results are presented in § 3. Of course, the particular bilinear transformation used here is not essentially different from that presented by Rosenbrock (1970) and McMillan (1952), and this link is formally established in § 4. Also presented in this section is an interpretation of a minimal basis as defined by Forney (1975).

In the sequel we shall use $\mathbb{C}$ to denote the field of complex numbers. If $\alpha(s)$, $\beta(s)$ are any two polynomials in $s$, with coefficients in $\mathbb{C}$, then we write $\alpha(s)|\beta(s)$ to denote that $\alpha(s)$ divides $\beta(s)$. Also $S(G(s))$ will be used to denote the Smith–McMillan form of $G(s)$ so that

$$S(G(s)) = \begin{cases} (Q(s),\, O_{m,\,l-m}) & m<l \\ Q(s) & m=l \\ \begin{pmatrix} Q(s) \\ O_{m-l,\,l} \end{pmatrix} & m>l \end{cases}$$

where,

$$Q(s) = \operatorname{diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\epsilon_p(s)}{\psi_p(s)}, 0, \ldots, 0\right)$$

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2 c 2

$p$ is the normal rank of $G(s)$ and $\epsilon_i|\epsilon_{i+1}$, $\psi_{i+1}|\psi_i$, $i = 1, \ldots, p-1$. Otherwise the reader is referred to Rosenbrock (1970) for the main results concerning polynomial system matrices and Pugh and Shelton (1978) for the definitions of the equivalence transformations denoted subsequently by (u.e.) and (e.u.e.).

## 2. Zeros and poles of a rational matrix

Let $D(s)$ be a polynomial matrix of dimension $m \times l$, then

*Definition* 1

$s_0 \in \mathbb{C}$ is a finite zero of degree $k$ of $D(s)$ in case $(s - s_0)^k$ is an elementary divisor of $D(s)$. The set of zeros of $D(s)$ is the set of all such numbers $s_0$, a zero of degree $k$ being included $k$ times.

It is noted that this is essentially Rosenbrock's (1974 b) definition wherein the zeros of $D(s)$ are the zeros of the invariant polynomials of $D(s)$ taken all together. The only new concept in Definition 1 is that of the degree of a zero. The following result yields a simple test to determine whether $s_0$ is a zero of $D(s)$ although it gives no information concerning the degree (the simple proof is omitted).

*Proposition* 1

$s_0 \in \mathbb{C}$ is a zero of $D$ if and only if,

$$\text{Rank } D(s_0) < \rho(D) \qquad (2.1)$$

where $\rho(\,.\,)$ denotes the normal rank of the indicated matrix.

Recall that if $G(s)$ is an $m \times l$ rational matrix then it may be decomposed into relatively prime polynomial factors,

$$G(s) = V_0 T_0^{-1} = T_c^{-1} U_c \qquad (2.2)$$

where $T_c$, $U_c$ are relatively left prime and $T_0$, $V_0$ are relatively right prime.

Neither of the factorizations (2.2) is unique since for any unimodular matrix $M(s)$,

$$T'_c = M T_c \quad V'_c = M V_c \qquad (2.3)$$

and

$$T'_0 = T_0 M \quad U'_0 = U_0 M \qquad (2.4)$$

also form relatively prime factorizations of $G(s)$. However, we set up the following terminology due to Wolovich (1974).

*Definition* 2

Any $m \times l$ polynomial matrix such as $U_0$ or $V_c$ satisfying (2.2) will be called a numerator of $G(s)$.

Any $m \times m$ polynomial matrix such as $T_0(s)$ or $l \times l$ matrix such as $T_c(s)$ satisfying (2.2) will be called a denominator of $G(s)$.

This definition is made more meaningful by the following result which establishes the complete connection between the different relatively prime factorizations of $G(s)$.

*Theorem* 1

All numerators of $G(s)$ are unimodular equivalent while all denominators are extended unimodular equivalent.

*Proof*

If $N_1(s)$ and $N_2(s)$ are two numerators of $G(s)$ and $D_1(s)$, $D_2(s)$ their denominators such that,

$$G(s) = D_1^{-1}(s)N_1(s) = D_2^{-1}(s)N_2(s)$$

then the result follows from Rosenbrock (1970, p. 139). In fact $D_1(s)$ and $D_2(s)$ are actually unimodular equivalent. The theorem is true in the same way in case,

$$G(s) = N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s)$$

If the factorizations are of the type,

$$G(s) = N_1(s)D_1^{-1}(s) = D_2^{-1}(s)N_2(s) \tag{2.5}$$

then,

$$D_2(s)N_1(s) = N_2(s)D_1(s) \tag{2.6}$$

However, from the coprimeness of the factorizations (2.5), (2.6) is simply the statement that $N_1(s)$ and $N_2(s)$ are extended unimodular equivalent, as indeed are $D_1(s)$ and $D_2(s)$. Since $N_1(s)$ and $N_2(s)$ are both $m \times l$ the relationship of extended unimodular equivalence may be replaced by one of unimodular equivalence (Pugh and Shelton 1978) which proves the theorem.

It is noted that the complete result concerning the numerators was originally proved by Wolovich (1974) although as indicated in the proof part of the result is due to Rosenbrock (1970). The complete result is of course inherent in the work of Fuhrmann (1977) but the simplicity of the above proof is a direct consequence of the results of Pugh and Shelton (1978). More particularly it follows from the same source that,

*Corollary* 1

All numerators (denominators) of a rational matrix $G(s)$ have the same set of zeros. Specifically if $s_0$ is a zero of degree $k$ of a numerator (denominator) of $G(s)$ it is a zero of degree $k$ of every numerator (denominator).

This corollary makes the following definition meaningful.

*Definition* 3

$s_0 \in \mathbb{C}$ is a zero (pole) of degree $k$ of the rational matrix $G(s)$ if it is a zero of degree $k$ of any numerator (denominator).

We now wish to define the ' infinite ' zeros and poles of $G(s)$. To accomplish this we utilize definition (3) and the standard technique of complex variable theory as has also been suggested by Verghese *et al.* (1978).

*Definition* 4

$G(s)$ is said to have an infinite zero (pole) of degree $k$ in case $w = 0$ is a finite zero (pole) of degree $k$ for the rational matrix $G(1/w)$.

In the following section we explore the consequences of these definitions.

### 3. Results for rational matrices

The definitions given above allow many of the results of complex variable theory concerning rational functions to be generalized. The following is one such result.

*Theorem* 2

$s_0 \in \mathbb{C}$ is a pole of the rational matrix $G(s)$ if and only if for some $i$ and $j$,

$$\lim_{s \to s_0} g_{ij}(s) = \infty \tag{3.1}$$

*Proof*

If $s_0 \in \mathbb{C}$ is a pole of the rational matrix $G(s)$ then $(s - s_0)$ is a factor of some invariant polynomial in the denominator of $G(s)$. In particular if the Smith form of the denominator of $G(s)$ is,

$$S(D) = \text{diag}\ (\psi_r, \psi_{r-1}, \ldots, \psi_1) \tag{3.2}$$

where $r = l$ or $m$ depending upon the factorization of $G(s)$, then $(s - s_0)$ divides $\psi_i(s)$ for some $i$. By the divisibility properties of the $\psi_i(s)$ we then have that,

$$(s - s_0) | \psi_1(s) \tag{3.3}$$

But $\psi_1(s)$ is the least common denominator of elements of $G(s)$ (Rosenbrock 1970) and hence $(s - s_0)$ occurs in the denominator of at least one element $g_{ij}(s)$ of $G(s)$. Thus by the corresponding theorem of complex variable theory,

$$\lim_{s \to s_0} g_{ij}(s) = \infty \tag{3.4}$$

Conversely since $g_{ij}(s)$ is a rational function its only singularities are poles (finite or infinite). Thus if (3.4) holds for finite $s_0$ then $s_0$ is a finite pole of $g_{ij}(s)$. Consequently $(s - s_0) | \psi_1(s)$ and hence $s_0$ is a pole of $G(s)$.

The following theorem generalizes another well known result concerning rational functions.

*Theorem* 3

The rational matrix $G(s)$ is polynomial if and only if it has no finite poles.

*Proof*

If $G(s)$ is polynomial then it has a coprime factorization,

$$G(s) = I_m^{-1}\, G(s).$$

Hence $I_m$ is a denominator of $G(s)$, and clearly $G(s)$ has no finite poles.

Conversely suppose that

$$G(s) = D^{-1}(s) N(s)$$

is a coprime factorization of $G(s)$. If $G(s)$ has no finite poles then all denominators of $G(s)$ have no finite zeros. Thus the Smith form of $D(s)$ is $I_m$, and so $D(s)$ is unimodular. Consequently $D^{-1}(s)$ is a polynomial matrix as is $G(s)$.

*Corollary* 2

A polynomial matrix has all its poles at infinity.

*Proof*

Obvious.

*Corollary* 3

If $G(s)$ is $m \times m$ with $\rho(G) = m$ then it is a unimodular polynomial matrix if and only if it has no finite poles and no finite zeros.

*Proof*

If $G(s)$ has no finite poles then by the above theorem it is polynomial. Since the polynomial matrix $G(s)$ has no finite zeros then its Smith form is $I_m$. i.e. $G(s)$ is unimodular.

Conversely if $G(s)$ is polynomial then it has no finite poles. Also since $G(s)$ is unimodular,

$$G(s) = [G(s)^{-1}]^{-1} I_m$$

is a prime factorization of $G(s)$ from which it is obvious that $G(s)$ has no finite zeros.

*Theorem* 4

A rational matrix $G(s)$ is proper if and only if it has no infinite poles.

*Proof*

Suppose $G(s)$ is proper, then each $g_{ij}(s)$ is a proper rational function, i.e.

$$\lim_{s \to \infty} g_{ij}(s)$$

exists for all $i$ and $j$. Thus in each $g_{ij}(s)$, the degree of the numerator is at most equal to the degree of the denominator. Hence in $g_{ij}(1/w)$, $w$ is not a factor of the denominator. Consequently $w$ is not a factor in the least common denominator of elements of $G(1/w)$. But this least common denominator is $\tilde{\psi}_1(w)$ the last invariant polynomial in the Smith form of the denominators of $G(1/w)$. By the divisibility properties of the invariant polynomials, $w$ cannot be a factor of any invariant polynomial of the denominators of $G(1/w)$. Thus $G(s)$ has no infinite poles.

Conversely suppose that $G(s)$ has no infinite poles, then $G(1/w)$ has no poles at $w = 0$. Thus $w$ is not a factor of any invariant polynomial of the denominators of $G(1/w)$. Particularly $w$ is not a factor in $\tilde{\psi}_1(w)$, the last such invariant polynomial. But $\tilde{\psi}_1(w)$ is the least common denominator of elements of $G(1/w)$ and so $w$ is not a factor in the denominator of any $g_{ij}(1/w)$. Thus for all $i$, $j$,

$$g_{ij}\left(\frac{1}{w}\right) = \frac{a_\tau + a_{\tau-1}w + \ldots + a_0 w^\tau}{b_\rho + b_{\rho-1}w + \ldots + b_0 w^\rho} \tag{3.5}$$

where,

$$b_\rho \neq 0 \text{ and } a_0 \neq 0 \tag{3.6}$$

for $\tau = \tau(i, j)$ and $\rho = \rho(i, j)$.

In view of the way in which (3.5) has been constructed from $g_{ij}(s)$ it follows from (3.5) and (3.6) that in $g_{ij}(s)$ the degree of the numerator is at most equal to the degree of the denominator. Thus $g_{ij}(s)$ is a proper rational function for all $i$, $j$. i.e. $G(s)$ is a proper rational matrix.

*Corollary* 4

A rational matrix $G(s)$ has infinite poles if and only if it is non-proper.

*Proof*

Immediate.

The next corollary extends Theorem 2 to the case of infinite poles.

*Corollary* 5

$G(s)$ has a pole at infinity if and only if for some $i$, $j$,

$$\lim_{s\to\infty} g_{ij}(s) = \infty.$$

*Proof*

Immediate.

It is well known that a rational function cannot simultaneously have poles and zeros at any $s_0 \in \mathbb{C}$, nor indeed at infinity. Thus a polynomial function has no infinite zeros, while a proper rational function has no infinite poles. In the case of rational matrices however it is perfectly possible that $s_0 \in \mathbb{C}$ may be both a pole and zero (in fact there may be many poles and zeros of differing degrees at $s_0$). In particular it is possible that a polynomial matrix may have infinite zeros. The next result provides for their absence.

*Theorem* 5

Let $G(s)$ be an $m \times l$ polynomial matrix of full rank. $G(s)$ has no infinite zeros if and only if there exists a high-order minor ($m \times m$ or $l \times l$ whichever is the less) of $G(s)$ with degree $\delta(G)$, where $\delta(G)$ denotes the McMillan degree of $G(s)$.

*Proof*

Assume that $m \leqslant l$, the other cases may be proved similarly.

Since $G(s)$ is polynomial it has no finite poles. Also $G(s)$ has a minor of degree $\delta(G)$ and (Rosenbrock 1970, p. 137)

$$\delta(G) = \nu\left(G\left(\frac{1}{w}\right)\right) \tag{3.7}$$

where $\nu(\,.\,)$ denotes the least order of the indicated matrix.

Let,

$$G\left(\frac{1}{w}\right) = \tilde{D}^{-1}(w)\tilde{N}(w) \tag{3.8}$$

be a prime factorization of $G(1/w)$ then by (3.7)

$$\delta(|\tilde{D}(w)|) = \delta(G) \tag{3.9}$$

where $|\,.\,|$ denotes the determinant of the indicated matrix.

Suppose $G(s)$ has an $m\times m$ minor of degree $\delta(G)$. Denote this minor by $G_{(j_1,j_2,\ldots,j_m)}$ thus indicating the columns from which it is formed. Now,

$$G_{(j_1,\ldots,j_m)}(s)\Big|_{s=1/w}=|\tilde{D}^{-1}(w)|\,.\,\tilde{N}_{(j_1,\ldots,j_m)}(w)$$

$$=\frac{1}{w^{\delta(G)}}.\,\tilde{N}_{(j_1,\ldots,j_m)}(w)$$

Hence

$$\tilde{N}_{(j_1,\ldots,j_m)}(w)=w^{\delta(G)}\,.\,G_{(j_1,\ldots,j_m)}(s)\Big|_{s=1/w}\tag{3.10}$$

Now

$$G_{(j_1,\ldots,j_m)}(s)=p_{\delta(G)}s^{\delta(G)}+\ldots+p_1s+p_0\tag{3.11}$$

with

$$p_{\delta(G)}\neq 0\tag{3.12}$$

and so

$$G_{(j_1,\ldots,j_m)}\Big|_{s=1/w}=\frac{p_{\delta(G)}+\ldots+p_1w^{\delta(G)-1}+p_0w^{\delta(G)}}{w^{\delta(G)}}\tag{3.13}$$

Thus from (3.10) and (3.13)

$$\tilde{N}_{(j_1,\cdots,j_m)}(w)=p_{\delta(G)}+\ldots+p_1w^{\delta(G)-1}+p_0w^{\delta(G)}.\tag{3.14}$$

Since $p_{\delta(G)}\neq 0$ it follows that $w$ does not divide $\tilde{N}_{(j_1,\ldots,j_m)}(w)$ and so it does not divide the greatest common divisor of the $m\times m$ minors of $\tilde{N}(w)$. But this greatest common divisor is just the product of the invariant polynomials of $\tilde{N}(w)$, and so $\tilde{N}(w)$ has no elementary divisors of the form $w^q$ $(q>0)$. i.e. $\tilde{N}(w)$ has no zeros at $w=0$ i.e. $G(s)$ has no infinite zeros.

Conversely suppose that $G(s)$ has no infinite zeros, then if (3.8) is a prime factorization of $G(1/w)$,

$$\text{Rank }\tilde{N}(0)=m$$

since $m\leqslant l$. Consequently there exists some $m\times m$ minor of $\tilde{N}(w)$, say $\tilde{N}_{(j_1,\ldots,j_m)}(w)$ which is not divisible by $w$.
i.e.

$$\tilde{N}(_{j_1,\ldots,j_m})(w)=n_\tau w^\tau+\ldots+n_1w+n_0\tag{3.15}$$

for some $\tau$ and,

$$n_0\neq 0.\tag{3.16}$$

Now since the factorization (3.8) is prime it follows that,

$$\delta(|\tilde{D}(w)|)=\delta(G).$$

and so,

$$G_{(j_1,\ldots,j_m)}(s)\Big|_{s=1/w}=\frac{1}{w^{\delta(G)}}\tilde{N}_{(j_1,\ldots,j_m)}(w)$$

$$=\frac{n_\tau w^\tau+\ldots+n_1w+n_0}{w^{\delta(G)}}\tag{3.17}$$

Thus,

$$G_{(j_1,\ldots,j_m)}(s)=[n_\tau(1/s)^\tau+\ldots+n_1(1/s)+n_0]s^{\delta(G)}$$

Now since $G(s)$ is polynomial $\tau \leqslant \delta(G)$, in fact from (3.16) it follows that,

$$\delta(G_{j_1,\ldots,j_m}(s)) = \delta(G).$$

Thus a high-order minor of degree $\delta(G)$ exists, as required.

*Example* 1

Let,

$$G(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \tag{3.18}$$

then $G(s)$ is unimodular and it follows from Corollary 3 that $G(s)$ has no finite poles and no finite zeros. This can of course be verified directly from (3.18).

From Corollary 5 it follows that $G(s)$ has at least one pole at infinity while Theorem 5 indicates the existence of at least one infinite zero. To verify these observations notice that,

$$G(1/w) = \begin{bmatrix} 1 & 1/w \\ 0 & 1 \end{bmatrix}$$

and,

$$G(1/w) = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} w & 1 \\ 0 & 1 \end{bmatrix} \tag{3.19}$$

is a prime factorization of $G(1/w)$.

Hence,

$$\tilde{D}(w) = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix} \tag{3.20}$$

is a denominator of $G(1/w)$ while

$$\tilde{N}(w) = \begin{bmatrix} w & 1 \\ 0 & 1 \end{bmatrix} \tag{3.21}$$

is a numerator.

Now $\tilde{D}(w)$ has a zero at $w = 0$ of degree 1, as does $\tilde{N}(w)$. Thus $G(s)$ has one infinite pole and one infinite zero.

*Example* 2

The condition in Theorem 5 that $G(s)$ be of full rank is important for consider,

$$G(s) = \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix} \tag{3.22}$$

Then by Theorem 3 $G(s)$ has no finite poles and since

$$S(G)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$G(s)$ has no finite zeros.

To consider the point at infinity we have

$$G\left(\frac{1}{w}\right)=\begin{bmatrix}1 & \frac{1}{w}\\ \frac{1}{w} & \frac{1}{w^2}\end{bmatrix}$$

and,

$$G\left(\frac{1}{w}\right)=\begin{bmatrix}0 & w\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -w & w^2\end{bmatrix}^{-1} \tag{3.23}$$

is a prime factorization of $G(1/w)$. Hence

$$\tilde{D}(w)=\begin{bmatrix}1 & 0\\ -w & w^2\end{bmatrix} \tag{3.24}$$

is a denominator of $G(1/w)$ while,

$$\tilde{N}(w)=\begin{bmatrix}0 & w\\ 0 & 1\end{bmatrix} \tag{3.25}$$

is a numerator.

It is then clear from (3.24) that $G(s)$ has an infinite pole of degree two and from (3.25) that $G(s)$ has no infinite zeros.

Thus although there is no high-order minor of degree $\delta(G)=2$, still $G(s)$ has no infinite zeros.

## 4. The McMillan degree and a characterization of minimal bases

Rosenbrock (1970) has also defined the poles and zeros (both finite and infinite) of an $m\times l$ rational matrix $G(s)$. In the finite case these definitions coincide with Definition 3, in the infinite case however Rosenbrock gave the following.

*Definitions*

(i) If any element of $G(s)$ tends to infinity as $s\to\infty$, then $G(s)$ is said to have a pole at infinity.

(ii) If every minor of some given order $k$ tends to zero as $s\to\infty$ then $G(s)$ is said to have a zero at infinity.

As can be seen from Corollary 5 this definition of infinite poles corresponds to that given in Definition 4. However, while Definition 5(ii) may be a sufficient condition for an infinite zero to exist it is not necessary as we will show below.

Rosenbrock (1970) has given an alternative characterization of infinite poles and zeros.

*Definition* 6

Let

$$s = \frac{\alpha p}{p-1} \tag{4.1}$$

where $\alpha$ is a constant which is not a finite pole or zero of a minor of any order of $G(s)$. Then $G(s)$ has an infinite pole if and only if $G[\alpha p/(p-1)]$ has a pole at $p=1$, and similarly $G(s)$ has an infinite zero in case $G[\alpha p/(p-1)]$ has a zero a $p=1$. This definition is independent of $\alpha$ subject only to the given condition.

The various definitions are illustrated in the following example.

*Example* 3

Let

$$G(s) = \begin{bmatrix} s & 0 \\ 0 & \frac{1}{s} \end{bmatrix} \tag{4.2}$$

Now

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \tag{4.3}$$

is a coprime factorization and hence by Definition 3, $G(s)$ has a finite pole and zero at $s=0$, both of degree one.

In the case of infinite poles and zeros,

$$G\left(\frac{1}{w}\right) = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} \tag{4.4}$$

is a coprime factorization and hence from Definition 4 $G(s)$ has one infinite pole and one infinite zero both of degree one.

We note that $G(s)$ does have one element which tends to infinity as $s \to \infty$ and so Definition 5(i) predicts the existence of at least one pole at infinity. This is of course consistent with the findings of Definition 4. On the contrary however Definition 5(ii) does not predict the existence of the infinite zero for $G(s)$ since the only $2 \times 2$ minor of $G(s)$ is unity and not all $1 \times 1$ minors of $G(s)$ tend to zero as $s \to \infty$.

To illustrate the Definition 6 make the substitution (4.1) where $\alpha \neq 0$. Then,

$$G\left(\frac{\alpha p}{p-1}\right)=\begin{bmatrix} \dfrac{\alpha p}{p-1} & 0 \\ 0 & \dfrac{p-1}{\alpha p} \end{bmatrix} \tag{4.5}$$

The Smith–McMillan form of (4.5) is then,

$$\begin{bmatrix} \dfrac{1}{p(p-1)} & 0 \\ 0 & p(p-1) \end{bmatrix} \tag{4.6}$$

from which it is clear that $G[\alpha p/(p-1)]$ has both a pole and zero at $p=1$. Thus by the Definition 6 $G(s)$ has a pole and zero at infinity.

It is therefore seen that Definitions 4 and 6 lead to the same results, while it is clear that Definition 5(ii) is not a necessary condition. In fact the equivalence of Definitions 4 and 6 may be proved directly.

*Theorem* 7

The Definitions 4 and 6 are equivalent.

*Proof*

It will be shown that the poles and zeros of $G(1/w)$ at $w=0$ occur in an identical manner to those of $G[\alpha p/(p-1)]$ at $p=1$.

Let a minor of $G(s)$ of some order be,

$$\phi(s)=\frac{a_q s^q + a_{q-1}s^{q-1}+\ldots+a_1 s+a_0}{b_q s^q + b_{q-1}s^{q-1}+\ldots+b_1 s+b_0} \tag{4.7}$$

where the numerator and denominator have no common factors and, $a_q$, $b_q$ are not simultaneously zero.

Let $\psi_1(w)$ denote the identical minor to $\phi(s)$ formed from $G(1/w)$ and $\psi_2(p)$ denote that formed from $G[\alpha p/(p-1)]$. Then,

$$\psi_1(w)=\phi\left(\frac{1}{w}\right)=\frac{a_q + a_{q-1}w+\ldots+a_1 w^{q-1}+a_0 w^q}{b_q + b_{q-1}w+\ldots+b_1 w^{q-1}+b_0 w^q} \tag{4.8}$$

where the numerator and denominator have no common factors since $s=1/w$ is one-to-one.

Similarly

$$\psi_2(p)=\phi\left(\frac{\alpha p}{p-1}\right)$$

$$=\frac{a_q(\alpha p)^q + a_{q-1}(\alpha p)^{q-1}(p-1)+\ldots+a_1\alpha p(p-1)^{q-1}+a_0(p-1)^q}{b_q(\alpha p)^q + b_{q-1}(\alpha p)^{q-1}(p-1)+\ldots+b_1\alpha p(p-1)^{q-1}+b_0(p-1)^q} \tag{4.9}$$

where the numerator and denominator again have no common factors since $s=\alpha p/(p-1)$ is one-to-one.

From (4.8) and (4.9) it is clear that $\psi_1(w)$ has a zero of degree $k$ (respectively pole of degree $k$) at $w=0$ if and only if $\psi_2(p)$ has a zero of degree $k$ (respectively pole of degree $k$) at $p=1$.

Thus a determinantal divisor (Rosenbrock and Storey 1970) of $G(1/w)$ will possess a factor of the form $w^h$ ($h$ an integer) if and only if the corresponding determinantal divisor of $G[\alpha p/(p-1)]$ possesses a factor of the form $(p-1)^h$. In view of this any invariant polynomial of $G(1/w)$ possesses a factor of the form $w^h$ if and only if the corresponding invariant polynomial of $G[\alpha p/(p-1)]$ possess a factor of the form $(p-1)^h$, which proves the theorem.

*Corollary* 6

If $\delta(G)$ denotes the McMillan degree of $G(s)$ then $\delta(G)$ is unchanged by the bilinear transformations of Definitions 4 and 6, and $\delta(G)$ represents the total number of poles (finite and infinite) of $G(s)$ counted according to their multiplicity and their degree.

*Proof*

This basic result is proved in Rosenbrock (1970) while the complete result is a consequence of Theorem 7.

In a similar way the following result is true.

*Corollary* 7

If $G(s)$ is square and non-singular over the field of rational functions then,

$$\delta(G)=\delta(G^{-1}). \tag{4.10}$$

Thus Corollary 7 indicates that in the case of a square invertible $G(s)$ the total number of poles of $G(s)$ is equal to the total number of poles of $G^{-1}(s)$. In fact rather more than this can be said, and the following result in some ways is a generalization of Desoer and Schulman (1974, Theorem 4).

*Theorem* 8

If $G(s)$ is square and invertible then the finite (respectively infinite) poles of $G(s)$ are the finite (respectively infinite) zeros of $G^{-1}(s)$, and the finite (respectively infinite) zeros of $G(s)$ are the finite (respectively infinite) poles of $G^{-1}(s)$.

*Proof*

Let $G(s)$ have Smith form,

$$S(G)=\begin{bmatrix} \dfrac{\epsilon_1(s)}{\psi_1(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\epsilon_m(s)}{\psi_m(s)} \end{bmatrix} \tag{4.11}$$

where $\epsilon_i|\epsilon_{i+1},\ \psi_{i+1}|\psi_i$ $(i=1, 2, \ldots, m-1)$.

Thus,

$$G(s) = L(s)S(G)R(s)$$

where $R(s)$ and $L(s)$ are unimodular matrices and so

$$G^{-1}(s) = R^{-1}(s)[S(G)]^{-1}L^{-1}(s).$$

Thus $G^{-1}(s)$ and $[S(G)]^{-1}$ have the same Smith–McMillan form namely

$$S(G^{-1}) = \begin{bmatrix} \dfrac{\psi_m(s)}{\epsilon_m(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\psi_1(s)}{\epsilon_1(s)} \end{bmatrix} \tag{4.12}$$

This proves the finite case of the theorem.

In the case of the poles and zeros at infinity we may carry out an analogous argument concerning the matrix $G(1/w)$, then noting that,

$$\left[G\left(\frac{1}{w}\right)\right]^{-1} = G^{-1}\left(\frac{1}{w}\right) \tag{4.13}$$

yields the theorem.

*Corollary* 8

If $G(s)$ is square and invertible then the total number of poles of $G(s)$ is equal to the total number of zeros.

*Proof*

This follows directly from Theorem 8 and Corollary 7.

The ideas of §§ 2 and 3 lend themselves to an interpretation of the work of Forney (1975). Forney viewed a basis for a rational vector space over the field of rational functions as being a rational matrix whose columns or rows are linearly independent. From such a basis a polynomial basis (consisting solely of polynomial vectors) may always be constructed. Forney then defined a minimal basis to be any polynomial basis satisfying additionally that

$$\left.\begin{array}{l} \text{(i)}: \ G(s) \text{ has full rank for all finite } s \in \mathbb{C}. \\ \text{(ii)}: \ \text{the high-order coefficient matrix of } G(s) \text{ has full rank.} \end{array}\right\} \tag{4.14}$$

In view of this we may say,

*Theorem* 9

If the rational matrix $G(s)$ forms a minimal basis then it has no finite poles and no finite or infinite zeros.

*Proof*

Since $G(s)$ is a minimal basis it is polynomial and hence has no finite poles by Theorem 3.

Also $G(s)$ satisfies (4.14) (i) and so by Proposition 1 $G(s)$ has no finite zeros.

Finally $G(s)$ satisfies (4.14) (ii) and so there exists a high-order minor whose degree is equal to the sum of the row or column (depending on the dimensions of $G(s)$) degrees. But $\delta(G)$ is the least upper bound on the degrees of all minors $G(s)$, and hence $G(s)$ satisfies the conditions of Theorem 5. Thus $G(s)$ has no infinite zeros.

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# Infinite frequency interpretations of minimal bases

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The relevance of minimal bases in infinite frequency considerations is shown to lie more in the aspect of factorizations of rational matrices than in their characterization of the absence of zeros (finite and infinite). Accordingly the paper distinguishes between relatively prime and minimal factorizations and establishes an important structural property of the latter. This property is demonstrated to be extremely useful in a study of the effect of constant output feedback on the infinite poles and zeros of a transfer function matrix.

## 1. Introduction

In this paper a definition of infinite poles and zeros is adopted which is consistent with that used for their finite counterparts. The basic procedure originally proposed by McMillan (1952), has since been discussed by other authors (Rosenbrock 1970, Verghese 1978, Pugh and Ratcliffe 1979, Vardulakis 1979). With this definition it is shown that row (or column) properness (Wolovich 1974) does not possess the interpretation of relative primeness at infinity or rather of a lack of zeros at infinity which has been ascribed to it elsewhere using an alternative definition (Anderson and Bitmead 1978). Nevertheless it is established that row (or column) properness does have an important structural implication for factorizations of rational matrices.

Specifically the paper distinguishes between the usual relatively prime factorizations and what are termed minimal factorizations of a given rational matrix $G(s)$. Relatively prime factorizations by definition display the finite pole-zero structure of $G(s)$ and minimal factorizations are demonstrated to possess an important structural property in their ability to display the pole-zero structure of $G(s)$ at all frequencies, both finite and infinite. This property is shown to be useful in the investigation of the effect of constant output feedback on the poles and zeros of a transfer function matrix. Of course the case of finite poles and zeros is well documented and in this note a pleasing extension of the result concerning the invariance of the finite zeros (Rosenbrock 1970) is obtained.

## 2. Preliminaries

We define the infinite poles and zeros of a rational matrix in a manner that is a direct extension of the scalar case of a rational function and consistent with the definition of finite poles and zeros (Wolovich 1973).

Recall therefore (Pugh and Ratcliffe 1979) that $s_0 \in \mathbb{C}$ (the set of complex numbers) is called a zero of degree $k$ of the $m \times l$ polynomial matrix $P(s)$ when $(s-s_0)^k$ is an elementary divisor of $P(s)$ (Gantmacher 1959, Rosenbrock 1970).

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Now any $m \times l$ rational matrix $G(s)$ may be decomposed into relatively prime polynomial factors, i.e.

$$G(s) = D_1(s)^{-1}N_1(s) = N_2(s)D_2(s)^{-1} \tag{1}$$

where $D_1(s)$ and $N_1(s)$ are relatively left prime and $N_2(s)$, $D_2(s)$ are relatively right prime. Any $m \times l$ polynomial matrix such as $N_1(s)$ or $N_2(s)$ in (1) is called a numerator while any $m \times m$ matrix $D_1(s)$ or $l \times l$ matrix $D_2(s)$ is called a denominator of $G(s)$. Appealing to the scalar case of rational functions we then define the finite zeros of $G(s)$ to be the finite zeros of any numerator, and the finite poles of $G(s)$ to be the finite zeros of any denominator.

It is not immediately clear that the above terminology is well-defined. However any difficulties may be resolved since it can be shown (Pugh and Shelton 1978, Pugh and Ratcliffe 1979) that all numerators have the same Smith form, while any two denominators have Smith forms which differ only by a trivial expansion. Consequently the above terms are completely defined and we refer to Pugh and Ratcliffe (1979) for a more complete discussion of this point.

In order to define the terms ' infinite pole ' and ' infinite zero ' we make use of the standard technique of complex variable theory and perform the bilinear transformation

$$s = 1/w \tag{2}$$

This transformation takes the point $s = \infty$ to the point $w = 0$ and the point $s = 0$ to the point $w = \infty$. All other points in the complex $s$-plane are carried onto finite points in the complex $w$-plane in a one–one manner. We thus say

*Definition* 1

The $m \times l$ rational matrix $G(s)$ possesses an *infinite zero* (respectively *pole*) of degree $k$ when $G(1/w)$ has a finite zero (resp. pole) of precisely that degree at $w = 0$.

As a consequence of this definition many of the results in complex variable theory concerning rational functions may be generalized to rational matrices. In this way polynomial matrices are characterized by the absence of finite poles and proper rational matrices by the fact they possess no infinite poles (Verghese 1978, Pugh and Ratcliffe 1979).

In the main the paper follows the notation and terminology of Rosenbrock (1970). The McMillan degree of a rational matrix $G(s)$ is denoted by $\delta(G)$ and it is noted that should $G(s)$ be a polynomial matrix then $\delta(G)$ is the highest degree for minors of $G(s)$ of all orders. If $G(s)$ is a polynomial vector then $\delta(G)$ is simply the highest degree among all elements of $G(s)$.

## 3. Relatively prime and minimal factorizations

Let $P(s)$ be an $m \times l$ polynomial matrix where to be definite we assume $m < l$. This assumption in no way restricts what is to be said, it being adopted merely for the purpose of the exposition. Suppose that the normal rank of $P(s)$, denoted $\rho(P)$, is $m$ and that the degree of row $i$ of $P(s)$ is $\delta_i$ $(i = 1, 2, \ldots, m)$. The high-order coefficient matrix of $P(s)$, denoted $[P]_h$, is that matrix whose $(i, j)$th element is the coefficient of $s^{\delta_i}$ in the $(i, j)$th element of $P(s)$. Recall (Forney 1975) that

*Definition* 2

The rows of $P(s)$ are (or simply, $P(s)$ is) said to form a *minimal basis* if

(i) $P(s)$ has full rank for all finite $s \in \mathbb{C}$, and
(ii) $[P]_h$ has full rank, i.e. $P(s)$ is row-proper.

The following result was established in Pugh and Ratcliffe (1979) but here we give a direct proof which is relevant to one of the main results of this paper.

*Theorem* 1

If the $m \times l$ polynomial matrix $P(s)$ forms a minimal basis then it possesses no finite and no infinite zeros.

*Proof*

From (i) of Definition 2 it follows that the Smith form of $P(s)$ is $(I_m, 0_{m\ l-m})$ and hence that $P(s)$ has no finite zeros.

Also from (ii) of Definition 2, $[P]_h$ has full rank. Let

$$\Lambda(s) = \operatorname{diag}(s^{\delta_1}, \ldots, s^{\delta_m}) \tag{3}$$

Then

$$P(1/w) = \Lambda(1/w)\tilde{P}(w) = \Lambda(w)^{-1}\tilde{P}(w) \tag{4}$$

is a polynomial factorization of $P(1/w)$ where

$$\tilde{P}(0) = [P]_h \tag{5}$$

Consider the matrix

$$(\Lambda(w),\ \tilde{P}(w)) \tag{6}$$

For $w = 0$

$$(\Lambda(0),\ \tilde{P}(0)) \equiv (0_{m\ m},\ [P]_h) \tag{7}$$

which evidently has full rank since $[P]_h$ has rank $m$. For any finite $w(\neq 0) \in \mathbb{C}$ the matrix (6) certainly has rank $m$ because of the form of $\Lambda(w)$. Hence $(\Lambda(w), \tilde{P}(w))$ has full rank for all finite $w \in \mathbb{C}$ and consequently (4) is a relatively prime polynomial factorization of $P(1/w)$.

It thus follows that $\Lambda(w)$ is a denominator and $\tilde{P}(w)$ a numerator of $P(1/w)$. In particular therefore the zeros of $P(1/w)$ at $w = 0$ are precisely the zeros at $w = 0$ of $\tilde{P}(w)$. However

$$\operatorname{rank} \tilde{P}(0) = \operatorname{rank} [P]_h = m$$

and so $\tilde{P}(w)$ has no zeros at $w = 0$. Hence from Definition 1, $P(s)$ has no infinite zeros, as required.

To see that the converse of this result does not hold consider the following example.

*Example* 1

Let

$$P_1(s) = \begin{pmatrix} s & 0 & 1 \\ s & 1 & 1 \end{pmatrix} \tag{8}$$

then $\rho(P_1)=2$. Thus $P_1(s)$ is a polynomial basis for a certain rational vector space.

In fact $P_1(s)$ has full rank for all finite $s \in \mathbb{C}$ but $P_1(s)$ is not minimal since

$$[P_1]_h = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{9}$$

which is clearly not of rank 2.

However consider the infinite zeros of $P_1(s)$. Now

$$\begin{aligned} P_1(1/w) &= \begin{pmatrix} 1/w & 0 & 1 \\ 1/w & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} w & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{10}$$

Now (10) is a relatively prime factorization of $P_1(1/w)$ since the matrix

$$\begin{pmatrix} w & 0 & 1 & 0 & w \\ -1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

has full rank for all finite $w \in \mathbb{C}$. Consequently

$$\begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \end{pmatrix}$$

is a numerator of $P_1(1/w)$, and this clearly has no zeros at $w=0$. Therefore $P_1(s)$ possesses no infinite zeros.

The above example demonstrates that according to the definition adopted here, row properness is not an exact characterization of polynomial matrices possessing no infinite zeros. We note however that in a recent paper Anderson and Bitmead (1978) have given an essentially different definition of the term ' infinite zeros ' under which row properness is an exact characterization of such matrices. Attractive as such a definition may appear from this point of view it is highly unsatisfactory from another. For under their definition the matrices $P_1(s)$ of Example 1 and

$$\begin{pmatrix} s & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{11}$$

do not have the same infinite zeros. However

$$P_1(s) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

so that $P_1(s)$ and (11) are only a simple constant transformation away from each other. Thus the infinite zeros as defined by Anderson and Bitmead (1978) are not invariant under such transformations. This difficulty arises because the construction used by Anderson and Bitmead introduces ' infinite zeros ' over and above those which are considered in this paper. It is these additional quantities which give rise to the situation just described and as a consequence represent dynamically uninteresting properties.

It is seen from Example 1 that row properness is a sufficient condition for the exclusion of infinite zeros in the sense of Definition 1, but not a necessary one. Thus from the point of view of the existence of infinite zeros the concept of row properness does not assume great importance. Despite this however the concept does possess a very interesting implication for relatively prime factorizations of rational transfer function matrices as is indicated by the following result.

*Theorem* 2

Let $G(s)$ be an $m \times l$ rational matrix and

$$G(s) = D(s)^{-1} N(s) \tag{12}$$

a polynomial factorization of $G(s)$ in which the matrix

$$(D(s),\ N(s)) \tag{13}$$

forms a minimal basis. Let the $i$th row degree of (13) be denoted by $\delta_i$ $(i = 1, \ldots, m)$ and define

$$\Lambda(s) = \operatorname{diag}\,(s^{\delta_1}, \ldots, s^{\delta_m}) \tag{14}$$

Then

(i) the finite poles of $G(s)$ are the finite zeros of $D(s)$ and the infinite poles of $G(s)$ are the zeros at $w = 0$ of the polynomial matrix

$$\Lambda(w) D(1/w) \tag{15}$$

(ii) the finite zeros of $G(s)$ are the finite zeros of $N(s)$ and the infinite zeros of $G(s)$ are the zeros at $w = 0$ of the polynomial matrix

$$\Lambda(w) N(1/w) \tag{16}$$

*Proof*

Since $(D(s),\ N(s))$ is a minimal basis the matrices $D(s)$ and $N(s)$ are relatively left prime by Definition 2 (i). Thus (12) is a relatively prime factorization of $G(s)$ and so the statement concerning the finite poles and zeros is immediate from their definition.

For the infinite case we have from Definition 2 (ii) that $[D,\ N]_h$ the high order coefficient matrix of (13) has full row rank. Hence, as in the proof of Theorem 1, there exists a relatively prime factorization

$$(D(1/w),\ N(1/w)) = \Lambda(w)^{-1} (\tilde{D}(w),\ \tilde{N}(w)) \tag{17}$$

in which the matrices $\tilde{D}(w)$ and $\tilde{N}(w)$ are themselves relatively left prime polynomial matrices and

$$\Lambda(w) = \operatorname{diag}\,(w^{\delta_1}, \ldots, w^{\delta_m}) \tag{18}$$

Now from (17)

$$\tilde{D}(w) = \Lambda(w) D(1/w), \quad \tilde{N}(w) = \Lambda(w) N(1/w) \tag{19}$$

and so

$$\tilde{D}(w)^{-1}\tilde{N}(w) = D(1/w)^{-1}\Lambda(w)^{-1}\Lambda(w)N(1/w) = D(1/w)^{-1}N(1/w)$$

Hence by (12)

$$\tilde{D}(w)^{-1}\tilde{N}(w) = G(1/w) \tag{20}$$

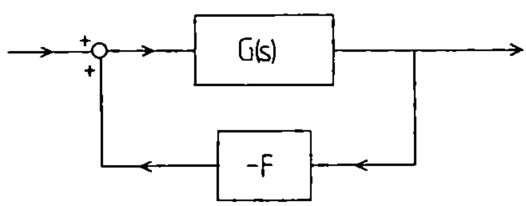
But $\tilde{D}(w)$ and $\tilde{N}(w)$ are relatively left prime polynomial matrices and so by Definition 1, the infinite poles of $G(s)$ are the zeros at $w=0$ of $\tilde{D}(w)$ and the infinite zeros of $G(s)$ are the zeros at $w=0$ of $\tilde{N}(w)$. By virtue of (19) the theorem follows as required.

It is thus seen that not only do the factorizations described in the theorem display the finite pole-zero structure of the underlying rational matrix (as indeed does any relatively prime factorization) but they additionally display the infinite pole-zero structure in a particularly simple way. This structural property is extremely useful in the feedback investigation of the following section. It therefore seems appropriate to distinguish between such factorizations of a given $G(s)$ and the usual relatively prime factorizations in the manner proposed by Forney (1975). Accordingly factorizations of the type described in Theorem 2 will be termed minimal.

We note finally that analogous conclusions may be drawn if $m > l$ if the terms ' rows ' and ' columns ' are interchanged, while if $m = l$ either of these terms may be used.

## 4. Output feedback considerations

The effect of constant output feedback on the finite poles and zeros of a transfer matrix has been considered by various authors and the results are well-documented. In particular it is known (Rosenbrock 1970) that the finite zero structure of the given rational matrix is completely unchanged by such action, although the finite pole structure possesses no such invariant property. It is of interest to know if this state of affairs persists when one considers the infinite poles and zeros.




Feedback system.

Accordingly let $G(s)$ be an $m \times l$ transfer function matrix and suppose that constant output feedback as summarized by the matrix $F$ is applied in the manner described in the Figure. If $G_F(s)$ denotes the transfer function matrix of the feedback system so constructed then

$$G_F(s) = G(s)(I + FG(s))^{-1} = (I + G(s)F)^{-1}G(s) \tag{21}$$

provided, as we will always assume, that

$$|I_l + FG(s)| \equiv |I_m + G(s)F| \neq 0 \tag{22}$$

The following result indicates that minimal factorizations of the open loop transfer function matrix $G(s)$ are closely related to minimal factorizations of the feedback transfer function matrix $G_F(s)$.

*Theorem* 3

If

$$G(s) = D(s)^{-1}N(s) \tag{23}$$

is a minimal factorization of $G(s)$ then

$$G_F(s) = (D(s) + N(s)F)^{-1}N(s) \tag{24}$$

is a minimal factorization of $G_F(s)$.

*Proof*

Substituting (23) into (21) gives

$$\begin{aligned} G_F(s) &= (I_m + D(s)^{-1}N(s)F)^{-1}D(s)^{-1}N(s) \\ &= (D(s) + N(s)F)^{-1}N(s) \end{aligned} \tag{25}$$

Hence (24) is certainly a polynomial factorization of $G_F(s)$. We must thus establish that this factorization is minimal.

Now

$$(D(s) + N(s)F,\ N(s)) = (D(s),\ N(s)) \begin{pmatrix} I_m & 0 \\ F & I_l \end{pmatrix} \tag{26}$$

Since $(D(s),\ N(s))$ forms a minimal basis, it has in particular full row rank for all finite $s \in \mathbb{C}$. Consequently from (26) it follows that $(D(s) + N(s)F,\ N(s))$ also has full row rank for all finite $s \in \mathbb{C}$, which of course implies that the factorization (25) is relatively prime.

Let

$$(d_i(s),\ n_i(s)) \tag{27}$$

denote the $i$th row of $(D(s),\ N(s))$ then the $i$th row of $(D(s) + N(s)F,\ N(s))$ is

$$(d_i(s) + n_i(s)F,\ n_i(s)) \tag{28}$$

If $\delta(n_i) < \delta(d_i)$ then the degree of (28) is clearly $\delta(d_i)$ since $F$ is a constant matrix. But in this case $\delta(d_i)$ is the degree of (27). Hence (27) and (28) have the same degree.

If on the other hand $\delta(n_i) \geqslant \delta(d_i)$ then since $F$ is constant

$$\delta(d_i + n_i F) \leqslant \delta(n_i) \tag{29}$$

However the last $l$ columns of (28) are still $n_i(s)$ and so the degree of (28) is precisely $\delta(n_i)$. But $\delta(n_i)$ in this case is the degree of (27) and so again (27) and (28) have the same degree.

It thus follows that $(D(s), N(s))$ and $(D(s)+N(s)F, N(s))$ have the same row degrees. Consequently from (26) we have

$$[D+NF, N]_h = [D, N]_h \begin{pmatrix} I_m & 0 \\ F & I_l \end{pmatrix} \tag{30}$$

Now $(D(s), N(s))$ is a minimal basis and so from Definition 2 (ii) its high order coefficient matrix $[D, N]_h$ has full rank. Hence from (30) the high order coefficient matrix of $(D(s)+N(s)F, N(s))$ has full rank and so

$$(D(s)+N(s)F, N(s)) \tag{31}$$

forms a minimal basis, as required.

This result in view of Theorem 2 gives immediately

*Theorem* 4

Let $\delta_i$ $(i=1, \ldots, m)$ denote the row degrees of

$$(D(s), N(s))$$

where

$$G(s) = D(s)^{-1}N(s)$$

is a minimal factorization, and let

$$\Lambda(w) = \text{diag}\,(w^{\delta_1}, \ldots, w^{\delta_m})$$

Then

(i) the finite poles of $G_F(s)$ are the finite zeros of $D(s)+N(s)F$ and the infinite poles of $G_F(s)$ are the zeros at $w=0$ of $\Lambda(w)(D(1/w)+N(1/w)F)$ ;

(ii) the finite zeros of $G_F(s)$ are the finite zeros of $N(s)$ and the infinite zeros of $G_F(s)$ are the zeros at $w=0$ of $\Lambda(w)N(1/w)$.

On noting from Theorems 3 and 4 that $N(s)$ is a numerator from minimal factorizations of both $G(s)$ and $G_F(s)$ we thus obtain

*Theorem* 5

The finite and infinite zeros of a transfer function matrix are invariant under constant output feedback.

The result for the finite zeros was originally proved by Rosenbrock (1970) and the above theorem represents a pleasing generalization of this result to include the infinite zeros of $G(s)$.

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# Feedback and The Poles and Zeros of the Transfer Function Matrix

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ABSTRACT

This report considers the effect of constant output feedback on the finite and infinite poles and zeros of the transfer function matrix and on the system decoupling zeros. The conditions under which the resulting feedback transfer function matrix is proper are also established.

## I. INTRODUCTION

The effect of constant output feedback on the finite poles and zeros of a transfer function matrix and on the finite decoupling zeros has been considered by various authors and the results are well documented. (See for example [5,7]). Recently attention has been focussed on defining the infinite poles and zeros of a rational matrix [1,10,11]. These definitions can be applied to system matrices and in this report various results concerning the effect of output feedback on both the finite and infinite poles and zeros of a transfer function matrix and on the finite and infinite decoupling zero are established.

In this report the main definitions are recalled in Section II. In Section III it is shown that the transfer function zeros are invariant whereas in Section IV it is found that the infinite decoupling zeros are not in general invariant although the finite decoupling zeros are invariant under the introduction of constant output feedback. In Section V a wellknown result due to Rosenbrock and Hayton [7] concerning the McMillan degree of the transfer function matrix is interpreted in terms of the finite and infinite poles of that matrix, and the conditions under which a proper transfer function matrix results after the introduction of constant output feedback are established.

## II. PRELIMINARIES

We define the poles and zeros of a rational matrix in a manner which is a direct extension of the case for rational functions. Recall that any $s_o \varepsilon \mathbb{C}$ is called a zero of degree k of the $m \times \ell$ polynomial matrix P(s) in case $(s-s_o)^k$ is an elementary divisor of P(s). [1,2,3]

Now any $m \times \ell$ rational matrix G(s) may be decomposed into relatively prime polynomial factors

$$G(s) = D_2^{-1}(s)\ N_2(s) = N_1(s)\ D_1^{-1}(s) \tag{1}$$

where $N_1(s)$ and $D_1(s)$ are relatively right prime and $D_2(s)$, $N_2(s)$ relatively left prime. Any $m \times \ell$ matrix such as $N_2(s)$ or $N_1(s)$ in (1) is called a numerator of G(s) while any $m \times m$ matrix $D_2(s)$ or $\ell \times \ell$ matrix $D_1(s)$ is called a denominator of G(s). Appealing to the scalar case of rational functions we then define the finite zeros of G(s) to be the finite zeros of any numerator and the finite poles of G(s) to be the finite zeros of any denominator. For the infinite frequences we make use of a standard technique of complex variable theory and make the transformation $s = 1/\omega$. This transformation takes the point $s = \infty$ to the point $\omega = 0$, and the point $s = 0$ to $\omega = \infty$. We thus define the infinite zeros of G(s) as the zeros at $\omega = 0$ of $G(1/\omega)$ and the infinite poles of G(s) as the poles at $\omega = 0$ of $G(1/\omega)$.

Whilst any factorisation of the form (1) will display the finite poles and zeros of G(s) the factorisation

$$G(s) = D^{-1}(s)\ N(s) \tag{2}$$

where (D(s), N(s)) forms a minimal basis displays both the finite and infinite poles and zeros of G(s) in a particularly simple way. A factorisation of this form is termed a minimal factorisation of G(s). The following theorem shows the importance of minimal factorisations:

Theorem 1

Let $G(s) = D^{-1}(s)\ N(s)$

with $(D(s), N(s))$ a minimal basis whose ith row degree is denoted by $\delta_i$. Let $\Lambda(\omega) = \text{diag}\ (\omega^{\delta_1}, \omega^{\delta_2}, \ldots, \omega^{\delta_n})$

then i) the finite poles of $G(s)$ are the finite zeros of $D(s)$ and the infinite poles of $G(s)$ are the zeros of $\Lambda(\omega)\ D(1/\omega)$ at $\omega = 0$.

ii) the finite zeros of $G(s)$ are the finite zeros of $N(s)$ and the infinite zeros of $G(s)$ are the zeros of $\Lambda(\omega)\ N(1/\omega)$ at $\omega = 0$.

Proof: This forms theorem 3 of reference [4].

Analogous conclusions can also be drawn for column proper factorisations of $G(s)$.

## III. THE EFFECT OF OUTPUT FEEDBACK ON THE TRANSFER FUNCTION ZEROS

Consider a system described by the rational transfer function matrix $G(s)$. A minimal factorisation of $G(s)$ may be written

$$G(s) = D^{-1}(s)\ N(s) \tag{3}$$

where $(D(s),\ N(s))$ (4)

is a minimal basis. Now apply constant output feedback F so that the transfer function matrix becomes

$$G_F(s) = G(s)(I + F\ G(s))^{-1} \tag{5}$$

$$= (I + G(s)\ F)^{-1}\ G(s) \tag{6}$$

provided $|I + F\ G(s)| \not\equiv 0$. In the following it is assumed that this condition is satisfied. Substituting for $G(s)$ from (3) in (5) gives

$$G_F(s) = [I + D^{-1}(s)\ N(s)F]^{-1}\ D^{-1}(s)\ N(s)$$

$$= [D(s) + N(s)\ F]^{-1}\ N(s) \tag{7}$$

Now $$[D(s) + N(s)\ F,\ N(s)] = [D(s),\ N(s)] \begin{bmatrix} I_m & 0 \\ F & I_e \end{bmatrix} \tag{8}$$

Hence, since $[D(s),\ N(s)]$ has full row rank for all finite s, $[D(s) + N(s)F,\ N(s)]$ also has full row rank and the factorisation (7) is a relatively prime factorisation of $G_F(s)$.

Now the row degrees of $[D(s) + N(s)\ F,\ N(s)]$ are exactly those of $[D(s),\ N(s)]$ since if $[d_i(s),\ n_i(s)]$ (9)

is the ith row of (4) then

$$[d_i(s) + n_i^T(s)\ F,\ n_i(s)] \tag{10}$$

is the ith row of (8). Consequently

(a) if $\delta(n_i) < \delta(d_i)$ the row degree of the ith row of (8) is equal to $\delta(d_i)$ which is the degree of the ith row of (4).

(b) if $\delta(n_i) \geqslant \delta(d_i)$ since the $n_i$ term in (10) remains unchanged, the degree of (10) is equal to $\delta(n_i)$ which is the degree of (9), i.e. the ith row degree of (8) is equal to the ith row degree of (4) for all i.

Further

$$\begin{bmatrix} \text{High order coefficienct} \\ \text{matrix of } (D(s) + N(s)F, N(s)) \end{bmatrix} = \begin{bmatrix} \text{High order coefficient} \\ \text{matrix of } (D(s), N(s)) \end{bmatrix} \times \begin{bmatrix} I_m & 0 \\ F & \ell_e \end{bmatrix} \quad (11)$$

Since (D,N) is a minimal basis the high order coefficient matrix of (4) has full rank. Hence (11) has full rank and consequently (7) is a minimal factorisation of $G_F(s)$.

These observations, in view of theorem 1, allow us to state the following theorem:

Theorem 2

i) the finite zeros of $G_F$ are the finite zeros of $N(s)$.

ii) the infinite zeros of $G_F$ are the zeros of $\Lambda(\omega)\ N(1/\omega)$ at $\omega = 0$ where $\Lambda(\omega)$ is defined as in theorem 1.

Corollary 1

The finite and infinite zeros of the transfer function matrix are invariant under constant output feedback.

The result for the finite zeros was originally proved by Rosenbrock [5], and the above theorem gives a generalisation of this result to include the infinite zeros of G(s).

## IV. THE EFFECT OF OUTPUT FEEDBACK ON THE DECOUPLING ZEROS

Consider the system described by the polynomial system matrix

$$S = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{12}$$

where $T(s)$, $U(s)$, $V(s)$, $W(s)$ are respectively $r \times r$, $r \times \ell$, $m \times m$ and $m \times \ell$. Now apply constant output feedback so that the system matrix becomes

$$S_F = \begin{bmatrix} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & -I & 0 \\ 0 & I & F & -I \\ 0 & 0 & I & 0 \end{bmatrix} \tag{13}$$

The relationship between the finite decoupling zeros of (12) and (13) is widely known and is stated in the following theorem:

Theorem 3

Let the system S have finite input decoupling zeros $\{\beta_i\}$, output decoupling zeros $\{\sigma_i\}$ and input-output decoupling zeros $\{\delta_i\}$. Then the system $S_F$ has

i) i.d. zeros $\{\beta_i\}$ and no others

ii) o.d. zeros $\{\sigma_i\}$ and no others

iii) i.o.d. zeros $\{\delta_i\}$ and no others.

Proof: This result is proved in [5] page 157.

The infinite input decoupling zeros of S are defined as the zeros at $\omega = 0$ of $(T(1/\omega)\ U(1/\omega))$ and the other infinite decoupling zeros are defined in a similar manner. The infinite decoupling zeros of $S_F$ are not identical to those of S as can be seen from the following example:

Example 1

$$\text{Let } S = \begin{bmatrix} s^3 & s & 1 & 1 \\ 1 & 0 & 0 & 0 \\ s^5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{14}$$

The input decoupling zeros of S are the zeros at $\omega = 0$ of

$$\begin{bmatrix} 1/\omega^3 & 1/\omega & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \omega^3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \omega^2 & \omega^3 & \omega^3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence S has 2 infinite input decoupling zeros.

Now apply output feedback $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$

The infinite input decoupling zeros of $S_F$ are the zeros at $\omega = 0$ of

$$\left[\begin{array}{cccccc|cc} 1/\omega^3 & 1/\omega & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\omega^5 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & f_{11} & f_{12} & -1 & 0 \\ 0 & 0 & 0 & 1 & f_{21} & f_{22} & 0 & -1 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|cc} \omega^2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \omega^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & f_{11} & f_{12} & 1 & 0 \\ 0 & 0 & 0 & -1 & f_{21} & f_{22} & 0 & 1 \end{array}\right] \begin{bmatrix} \omega^5 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & I_6 \end{bmatrix}^{-1}$$

This is a prime factorisation and the numerator has Smith form

$$\begin{pmatrix} I_5 & 0 & 0 \\ 0 & \omega^5 & 0 \end{pmatrix}$$

Hence the system with output feedback has 5 infinite input decoupling zeros, i.e. the introduction of constant output feedback has increased the number of infinite input decoupling zeros.

Although the infinite decoupling zeros are not in general invariant under constant output feedback there are certain special cases when the infinite decoupling zero structure is not altered.

Theorem 4

Let $P(s) = \begin{bmatrix} D(s) & N(s) \\ -I & 0 \end{bmatrix}$ (15)

be a realisation of the transfer function matrix

$$G(s) = D^{-1}(s)\, N(s) \tag{16}$$

When output feedback is applied the system matrix becomes

$$P_F(s) = \left[\begin{array}{ccc:c} D(s) & N(s) & 0 & 0 \\ -I & 0 & I & 0 \\ 0 & -I & F & I \\ \hdashline 0 & 0 & -I & 0 \end{array}\right] \tag{17}$$

Then $P(s)$ and $P_F(s)$ have the same decoupling zeros, both finite and infinite.

Proof: The result for the finite input, output and input/output decoupling zeros is a special case of theorem 3 i).

Let

$$(D(1/\omega),\ N(1/\omega)) = \tilde{D}^{-1}(D,N)\quad (\tilde{N}(D),\ \tilde{N}(N))$$

be a relatively prime factorisation of $(D(1/\omega),\ N(1/\omega))$. Hence the infinite input decoupling zeros of $P(s)$ are the zeros at $\omega = 0$ of $(\tilde{N}(D),\ \tilde{N}(N))$.

Now
$$\begin{bmatrix} D(1/\omega) & N(1/\omega) & 0 & 0 \\ -I & 0 & I & 0 \\ 0 & -I & F & I \end{bmatrix} = \begin{bmatrix} \tilde{D}^{-1}(D,N) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}(D) & \tilde{N}(N) & 0 & 0 \\ -I & 0 & I & 0 \\ 0 & -I & F & I \end{bmatrix}$$

This is a relatively prime factorisation. Consequently the infinite input decoupling zeros of $P_F(s)$ are the zeros at $\omega = 0$ of

$$\begin{bmatrix} \tilde{N}(D) & \tilde{N}(N) & 0 & 0 \\ -I & 0 & I & 0 \\ 0 & -I & F & I \end{bmatrix}$$

which can be reduced by constant transformations to the form

$$\begin{bmatrix} \tilde{N}(D) & \tilde{N}(N) & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

Hence the infinite input decoupling zeros of $P_F(s)$ are the zeros at $\omega = 0$ of

$$(\tilde{N}(D),\ \tilde{N}(N))$$

Clearly $P(s)$ has no infinite output decoupling zeros. The infinite output decoupling zeros of $P_F(s)$ are the zeros at $\omega = 0$ of

$$\begin{bmatrix} D(1/\omega) & N(1/\omega) & 0 \\ -I & 0 & I \\ 0 & -I & F \\ 0 & 0 & -I \end{bmatrix} = \begin{bmatrix} \tilde{D}(D,N) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}(D) & \tilde{N}(N) & 0 \\ -I & 0 & I \\ 0 & -I & F \\ 0 & 0 & -I \end{bmatrix}$$

This is a relatively prime factorisation and the numerator has full rank for all $\omega$. Hence $P_F(s)$ has no infinite output decoupling zeros.

Neither $P(s)$ nor $P_F(s)$ has any infinite input/output decoupling zeros since neither has any output decoupling zeros.

<u>Corollory 2</u>

If $P(s)$ as in (15) has $(D(s)\ N(s))$ a minimal basis then $P_F(s)$ has no finite nor infinite decoupling zeros.

Of course analogous statements hold concerning the decoupling zero structure of system matrices in the form

$$P_1(s) = \begin{bmatrix} D(s) & I \\ -N(s) & 0 \end{bmatrix}.$$

Any polynomial system matrix such as (12) can be realised in the state space form

$$S_S(s) = \begin{bmatrix} sI-A & B \\ -C & D(s) \end{bmatrix} \tag{18}$$

where A, B, C are constant matrices respectively n x n, n x $\ell$ and m x r and D(s) in polynomial. $S_S(s)$ has no infinite decoupling zeros. The next result shows the effect of constant output feedback on the decoupling zeros of a system matrix written in this form.

<u>Theorem 5</u>:

If a system matrix $S_S(s)$ in state space form has no infinite decoupling zeros, then infinite decoupling zeros cannot be introduced merely by the addition of constant output feedback.

<u>Proof</u>: With constant output feedback (18) becomes -

$$\left[\begin{array}{ccc|c} sI_n-A & B & 0 & 0 \\ -C & D(s) & -I & 0 \\ 0 & I & F & -I \\ \hline 0 & 0 & I & 0 \end{array}\right] \tag{19}$$

The infinite input decoupling zeros are the zeros at $\omega = 0$ of

$$\left[\begin{array}{ccc|c} 1/\omega I_n-A & B & 0 & 0 \\ -C & D(1/\omega) & -I_m & 0 \\ 0 & I_\ell & F & -I_\ell \end{array}\right] \tag{20}$$

$$= \begin{bmatrix} \omega I_n & 0 & 0 \\ 0 & \Upsilon(\omega) & 0 \\ 0 & 0 & I_\ell \end{bmatrix}^{-1} \left[\begin{array}{ccc|c} I_n-A\omega & B\omega & 0 & 0 \\ -\Upsilon(\omega)C & \mathcal{V}(\omega) & -\Upsilon(\omega) & 0 \\ 0 & I_\ell & F & I_\ell \end{array}\right] \tag{21}$$

$$= D_1^{-1}(\omega)\ N_1(\omega)$$

where $D(1/\omega) = \Upsilon^{-1}(\omega)\ \mathcal{V}(\omega)$ is a prime factorisation.

Hence $N_1(\omega)$ has full row rank for all $\omega$, (21) is a relatively prime factorisation and $N_1(\omega)$ has no zeros at $\omega = 0$. i.e. the system with constant output feedback has no infinite input decoupling zeros.

The analogous result for the infinite output decoupling zeros can be proved in a similar manner. Note that this result is independent of the value of F.

## V. THE EFFECT OF OUTPUT FEEDBACK ON THE TRANSFER FUNCTION POLES

Any rational function g(s) can be decomposed as

$$g(s) = g_s(s) + d(s) \tag{22}$$

where $g_s(s)$ is strictly proper and d(s) is polynomial. Then the finite poles of g(s) are the finite poles of $g_s(s)$ and the infinite poles of g(s) are the infinite poles of d(s). An analogous result can be derived for rational matrices, but first the following lemma is required.

### Lemma 1

Let $G(s) = N(s) + D(s)$ (23)

where G(s) is a rational matrix and D(s) is a polynomial matrix. Then N(s) + D(s) has the same finite poles as N(s).

Proof: Rosenbrock ([5] p. 113, exercise 4.1) states that N(s) has the same denominator polynomials $\psi_i$ in its McMillan form as G(s). The finite poles are the zeros of the $\psi_i$ and hence the lemma follows.

### Theorem 5

Let G(s) be a rational matrix. Then G(s) can be written

$$G(s) = G_S(s) + D(s) \tag{24}$$

where $G_S(s)$ is strictly proper and D(s) is polynomial. Then

i) the finite poles of G(s) are the finite poles of $G_S(s)$

ii) the infinite poles of G(s) are the infinite poles of D(s)

Proof:

i) obvious since $G_S(s)$ is a special case of N(s) in Lemma 1.

ii) let $s = 1/\omega$ so that

$$G(1/\omega) = G_S(1/\omega) + D(1/\omega)$$

Since $G_S(\ )$ is strictly proper it has no infinite poles and hence $G_S(1/\omega)$ has no poles at $\omega = 0$. Since D(s) is polynomial it has no finite poles and hence $D(1/\omega)$ has all its poles at $\omega = 0$.

Now let $G_S(1/\omega) = \frac{1}{d(\omega)} M(\omega)$ and hence $G(s) = \frac{1}{d(\omega)} [M(\omega) + D(1/\omega)\ d(\omega)]$

where $d(\omega)$ is the lowest common denominator of all the elements of $G_S(s)$ and $M(\omega)$ is polynomial. Note that $d(0) \neq 0$ and $[M(\omega) + D(1/\omega)d(\omega)]$ has no poles except at $\omega = 0$.

Now the infinite poles of $G(s)$ are the poles at $\omega = 0$ of $[M(\omega) + D(1/\omega)d(\omega)]$ and, by Lemma 1, these are the poles of $D(1/\omega)d(\omega)$ i.e. the infinite poles of $D(s)$ are the infinite poles of $G(s)$.

Kalman [6] defines the McMillan degree of a rational matrix $G(\ )$ according to the equation

$$\delta(G) = \nu(G_S(s)) + \nu(D(s^{-1})$$

where $G_S(s)$ and $D(s)$ are defined as in equation(24). From theorem 5 it can be seen that $\nu(G_S(s))$ represents the total number of finite poles of $G(s)$ and $\nu(D(s^{-1}))$ represents the total number of infinite poles of $G(s)$.

The following result follows directly from a widely known result concerning the effect of output feedback on the McMillan degree of the transfer function matrix.

Theorem 7

The total number of poles (both finite and infinite) of the transfer function matrix is invariant under output feedback although the number of finite poles may change.

Proof: Rosenbrock and Hayton [7] have shown that $\delta(G) = \delta(G_F)$ where $\delta(\cdot)$ denotes the McMillan degree of the indicated matrix. Since the total number of poles of a rational matrix is equal to its McMillan degree the total number of poles of $G$ is equal to the total number of poles of $G_F$. However, the least order of $G$ is not in general equal to that of $G_F$ and hence the number of finite poles may change when constant output feedback is applied.

The following example illustrates this theorem:

Example 2

Let $P(s) = \begin{bmatrix} s+1 & 1 \\ 1 & s \end{bmatrix}$

Hence $G(s) = \dfrac{s^2+s+1}{s+1}$ and $G(1/\omega) = \dfrac{1+\omega+\omega^2}{\omega(1+\omega)}$

i.e. $G(s)$ has one finite pole at $s = -1$ and 1 infinite pole.

Now apply output feedback so that $P(s)$ becomes

$$P_1(s) = \left[\begin{array}{ccc|c} s+1 & 1 & 0 & 0 \\ 1 & s & -1 & C \\ 0 & 1 & 1 & -1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right]$$

with $G_F(s) = \dfrac{s^2+s+1}{s^2+2s+2}$ and $G_F(\omega) = \dfrac{1+\omega+\omega^2}{1+2\omega+2\omega^2}$

i.e. $G_F(s)$ has 2 finite poles at $s = 1 \pm i$ and no infinite poles.

Hence the total number of poles has not been changed by the addition of constant output feedback although the number of finite poles has been changed.

This result shows that output feedback changes the position of the transfer function poles. It can transform finite poles to infinite poles and vice versa. Anderson and Scott [8] have shown that $G_F(s)$ is almost always proper, i.e. $G_F(s)$ almost always has only finite poles. The problem of determining proper transfer function matrices which are output feedback equivalent to a given rational matrix will now be investigated. Such matrices will be at most proper since.

Theorem 8

$G_F(s)$ is strictly proper if and only if $G(s)$ is strictly proper.

Proof: This is theorem 2.7, Chapter 2 in [9].

In case G(s) is not strictly proper recall that any transfer function matrix G(s) can be expanded as

$$G(s) = G_S(s) + D(s) \tag{25}$$

as in equation (24). A least order realisation of G(s) is

$$P(s) = \begin{bmatrix} sI_\nu - A & B \\ -C & D(s) \end{bmatrix}$$

Note that P(s) has no decoupling zeros and G(s) has McMillan degree

$$\delta(s) = \nu + \delta(D(s)) \tag{26}$$

where $\nu$ is the least order of G(s).

With the introduction of constant output feedback F the transfer function matrix becomes

$$G_F(s) = G(s)\ (I + FG(s))^{-1} \tag{27}$$

which can be realised as

$$P_F(s) = \left[\begin{array}{ccc|c} sI_\nu - A & B & 0 & 0 \\ -C & D(s) & I & 0 \\ 0 & -I & F & I \\ \hline 0 & 0 & -I & 0 \end{array}\right] \tag{28}$$

$P_F(s)$ can be reduced by constant transformations to the form

$$P'_F(s) = \left[\begin{array}{cc|c} sI_\nu - A & BF & B \\ -C & I+D(s)F & D(s) \\ \hline 0 & -I & 0 \end{array}\right] \tag{29}$$

Since P(s) had no finite decoupling zeros, by theorem 3, $P'_F(s)$ will also have no finite decoupling zeros, i.e. $P'_F(s)$ is a least order realisation of $G_F(s)$.

Hence $\nu\ (G_F) = n_F$

$$= \delta\left(\begin{bmatrix} sI_\nu - A & BF \\ -C & I+D(s)F \end{bmatrix}\right) \tag{30}$$

The conditions under which all the infinite poles of G(s) become finite poles in $G_F(s)$ can now be stated.

Theorem 9

For a given rational transfer function matrix G(s) expanded as in (24), a proper transfer function matrix $G_F(s)$ will result when constant output feedback F is applied if and only if

$$\delta(\ |I + D(s)F|\ ) = \delta(D(s)) \tag{31}$$

Proof: The determinant

$$\begin{vmatrix} sI_\nu - A & BF \\ -C & I+D(s)F \end{vmatrix} \tag{32}$$

can be expanded by the first $\nu$ rows using the Laplace expansion. Clearly the highest degree for determinants generated from the first $\nu$ rows of (32) is $\nu$. Further the highest degree among minors of all orders of

$$(-C \quad I+D(s)F) \tag{33}$$

is its McMillan degree, and

$$\delta((-C \ I+D(s)F)) = \delta(I+D(s)F) \tag{34}$$

Hence $\delta(D(s))$ is an upper bound for the degree of minors of all orders of (33).

Now if

$$\delta(\ |I+D(s)F|\ ) = \delta(D(s))$$

then the above Laplace expansion of (32) will contain a term of degree $\nu + \delta(D(s))$. From the form of (32) it follows that this is the only term which can possess this degree, in fact all other terms in the Laplace expansion have degree strictly less than $\nu + \delta(D(s))$.

Hence (32) has degree $\nu + \delta(D(s))$ if and only if (31) holds. Now if (32) has degree $\nu + \delta(D(s))$ then since $P_F'$ has least order

$$\begin{aligned} n_F = \nu_F &= \nu + \delta(D(s)) \\ &= \delta(G) \end{aligned}$$

from equation (26). However, by theorem 7, $\delta(G) = \delta(G_F)$ and so

$$\nu_F = \delta(G_F)$$

i.e. $G_F$ is proper

On the other hand if $G_F$ is proper then the above argument may be reversed to show that

$$\eta_F \ (=\nu_F) = \nu + \delta(D(s))$$

Since $\eta_F = \nu + \delta(D(s))$ if and only if (31) holds it therefore follows that $G_F$ is proper if and only if

$$\delta( |I + D(s)F| ) = \delta(D(s)).$$

Corollory 3

The feedback system has no infinite poles if and only if

$$\delta( |I + DF| ) = \delta(D(s))$$

As a consequence of this we see that the feedback system will possess infinite poles whenever

$$\delta( |I + DF| ) < \delta(D(s))$$

Corollory 4

Given a proper transfer function matrix

$$G(s) = G_S(s) + D$$

where D is constant and $G_S(s)$ is strictly proper, a non-proper transfer function matrix will result when constant output feedback F is applied if and only if

$$|I + DF| = 0 \qquad (35)$$

Proof: If D is constant then $\delta(G) = \nu$. (36)

Also $\eta_F = \nu_F$

$$= \delta\left[\begin{vmatrix} sI_\nu - A & BF \\ -C & I + DF \end{vmatrix}\right]$$

This determinant has degree $\nu$ if and only if $|I + DF| \neq 0$.

But, from theorem 7, $\delta(G) = \delta(G_F)$. Therefore, if (35) does not hold

$$\nu_F = \nu$$

$$= \delta(G)$$

$$= \delta(G_F)$$

i.e. $G_F$ is proper.

If $|I + DF| = 0$ then $\nu_F < \nu$ and $G_F$ is non-proper, having $\nu_F$ finite poles and $\nu-\nu_F$ infinite poles.

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